Natural labelling schemes for simple roots and irreducible representations of exceptional Lie algebras

1981 J. Phys. A: Math. Gen. 14 15

(http://iopscience.iop.org/0305-4470/14/1/006)

J. Phys. A: Math. Gen. **14** (1981) 15–49. Printed in Great Britain

Natural labelling schemes for simple roots and irreducible representations of exceptional Lie algebras

R C King†§ and A H A Al-Qubanchi‡

† Department of Mathematics, University of Auckland, New Zealand
‡ Department of Physics and Mathematics, College of Education, University of Basrah, Basrah, Iraq

Received 3 January 1980

Abstract. The simple root system of each exceptional, simple Lie algebra is explicitly constructed in a variety of forms. Each construction is based on the natural embedding in the exceptional algebra of a classical, semi-simple algebra of the same rank. The procedure adopted leads to the discovery of a number of new chains of subalgebras illustrated by means of supplemented Dynkin diagrams. The various sets of simple roots are then used to determine natural labels for the irreducible representations of each of the exceptional algebras. For each such labelling scheme the modification rules for dealing with non-standard representation labels are tabulated. The connection between the natural labels and Dynkin labels is given in detail and a comparison is made with the labels of Wybourne and Bowick.

1. Introduction

The classification of complex semi-simple Lie algebras was completed by Cartan (1894). The classification involves the four sequences of simple Lie algebras A_k, B_k, C_k and D_k associated with the classical Lie groups $SU(k+1)$, $SO(2k+1)$, $Sp(2k)$ and $SO(2k)$ respectively, and the five exceptional Lie algebras G_2, F_4, E_6, E_7 and E_8 associated with exceptional Lie groups denoted, here as elsewhere, by the same symbols as used for the algebras. In each case the subscript is a positive integer specifying the rank of the algebra.

Dynkin (1962, p 432) introduced the idea of simple roots in order to improve the procedure for deriving Cartan's classification scheme. The simple roots serve not only to label, through Dynkin diagrams, each complex semi-simple Lie algebra, but also when combined with a set of non-negative integers, one for each simple root, to label each irreducible representation of the Lie algebra and the associated Lie group. Such a label was defined by Dynkin (1957b, p 329) in terms of the highest weight of the irreducible representation which also serves to label that representation.

An alternative labelling scheme has been developed for the irreducible representations of the classical groups. This followed the pioneer work of Murnaghan (1938, 1958), Weyl (1939) and Littlewood (1940) and is well summarised by Wybourne (1970). Similarly related work by many authors on the irreducible representations of the exceptional Lie groups has culminated in the definitive study by Wybourne and

§ Permanent address: Department of Mathematics, University of Southampton, Southampton, England

Bowick (1977), who provided labelling schemes for the irreducible representations of each of the exceptional Lie groups.

What is of concern here is the relationship between the labelling schemes of Dynkin and the rather more natural schemes emphasising such things as the tensor and spinor properties of the irreducible representations of the classical groups and the nature of embedding of the classical groups in the exceptional groups. This latter aspect of the labelling scheme is quite crucial, as is made clear by both Dynkin (1957b, p 352) and Wybourne and Bowick (1977). In the present paper a systematic construction of natural labelling schemes is carried out.

In the following section the simple root system of each classical simple Lie algebra is given and a straightforward way of establishing the simple roots of each of the exceptional algebras is described. These simple roots are expressed in terms of vectors in a Euclidean space chosen so as to emphasise in each case some particular classical subalgebra.

These results are discussed in § 3 and extended with a view to making the significance of certain freedoms of choice apparent and to proving the existence of some previously overlooked chains of embeddings.

The application to labelling irreducible representations of the semi-simple Lie algebras is given in § 4. Tables are presented giving the precise link between Dynkin labels and natural labels analogous to the links given by Wybourne (1974, p 130) and Wybourne and Bowick (1977). Further to this the corresponding modification rules appropriate to non-standard, inadmissible natural labels are tabulated. These results generalise that of Littlewood (1940, p 98) for S-functions which are characters of the unitary groups. They are given in precisely that form which makes them of use in evaluating Kronecker products of irreducible representations by the method due to Racah (1964) and Speiser (1964).

2. Simple root systems

A complex semi-simple Lie algebra g, of rank k and dimension n, may be decomposed into a direct sum of subspaces: one consisting of the Cartan subalgebra of dimension k spanned by a set of mutually commuting generators of the corresponding real semi-simple Lie group, G, and the other spanned by the non-degenerate eigenvectors, in g, of these generators. The corresponding eigenvalues determine the set of roots, Σ_g, of the Lie algebra g. Each root in Σ_g is indexed by a label $\alpha = \pm 1, \pm 2, \ldots, \pm(n-k)/2$, and is a vector in a k-dimensional root space V.

The root space V may conveniently be treated as a subspace, not necessarily proper, of a Euclidean space W of dimension d with $d \geqslant k$. This space possesses the usual Kronecker metric δ_{ij} with $i, j = 1, 2, \ldots, d$. The basis vectors of W are mutually orthogonal unit vectors $\boldsymbol{e}_i$ with $i = 1, 2, \ldots, d$. These are defined so that the jth component of $\boldsymbol{e}_i$ is given by $(\boldsymbol{e}_i)_j = \delta_{ij}$. In this space W, the root vector indexed by α is denoted by $\boldsymbol{r}(\alpha)$ and has componenets $r_i(\alpha) = \boldsymbol{e}_i \cdot \boldsymbol{r}(\alpha)$. The orthogonal complement, $V^{\perp}$, of V in W is spanned by a set, Γ_g, of $(d-k)$ vectors $\boldsymbol{p}$ which are such that

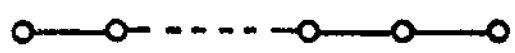
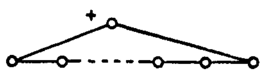
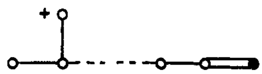
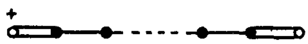
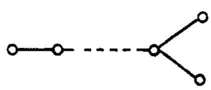
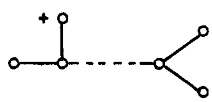
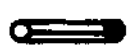
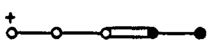
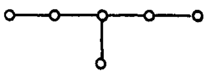
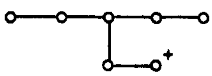
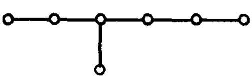
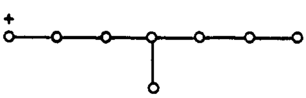
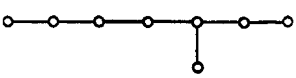
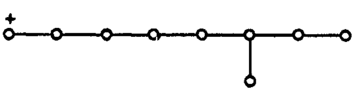
$$\boldsymbol{p} \cdot \boldsymbol{r}(\alpha) = 0 \tag{2.1}$$

for all roots $\boldsymbol{r}(\alpha)$.

An ordering of vectors in V may be introduced by means of an ordering in W defined in such a way that $\boldsymbol{v}$ is higher than $\boldsymbol{w}$, signified by $\boldsymbol{v} > \boldsymbol{w}$, if and only if the first non-vanishing component of $\boldsymbol{v}-\boldsymbol{w}$ is positive. A root vector $\boldsymbol{r}=\boldsymbol{r}(\alpha)$ is said to be positive if $\boldsymbol{r}>\boldsymbol{0}$ where all the components of $\boldsymbol{0}$ are zero.

Dynkin (1962, p 432) defined a root to be simple if it is positive and cannot be decomposed into the sum of two positive roots. The number of simple roots of a complex semi-simple Lie algebra g, of rank k, is precisely k, and Dynkin showed further (Dynkin 1962, p 461) that the system of simple roots, Π_g, of the algebra g defines that algebra up to isomorphism. Thus the classification of complex semi-simple Lie algebras is effected by enumerating all possible systems of simple roots. The results are conveniently summarised by means of the Dynkin diagrams of table 1, each of which labels unambiguously a complex simple Lie algebra g from which a unique compact real Lie group may be obtained by exponentiation. The standard notation due to Dynkin (1957b, p 365) adopted here is such that simple roots are represented by circles. Angles between simple roots of values $\pi/2$, $2\pi/3$, $3\pi/4$ and $5\pi/6$ are indicated by connecting the roots by 0, 1, 2 and 3 lines respectively. In the case of a simple Lie algebra, the corresponding simple roots are of at most two distinct lengths. If two distinct lengths do occur, the circles corresponding to simple roots of least length are filled whilst the others are left unfilled.

Table 1.

Lie group G	Lie algebra g	Dynkin diagram	Extended Dynkin diagram
$SU(k+1)$	A_k		
$SO(2k+1)$	B_k		
$Sp(2k)$	C_k		
$SO(2k)$	D_k		
G_2	G_2		
F_4	F_4		
E_6	E_6		
E_7	E_7		
E_8	E_8		

The key to the labelling of the irreducible representations both of the Lie algebra, g, and the corresponding Lie group, G, lies in the choice of W, along with its dimension d, and the precise specification of the simple roots with respect to the basis vectors $\boldsymbol{e}_i$ of W. For the classical Lie algebras A_k, B_k, C_k and D_k, labelling schemes are well established, based on spaces of dimension $k+1$, k, k and k respectively. The corresponding simple roots are given in table 2 along with, in the case of A_k, the single vector $\boldsymbol{p}$ spanning the complementary space $V^{\perp}$.

In the case of the exceptional Lie algebras, no such consensus of opinion regarding the specification of the simple roots has emerged. Thus the tabulations of Dynkin (1957b, p 378), Bourbaki (1968, p 202), Carter (1972, p 46), Gruber and Samuel (1975, p 105) and Wan (1975, p 99) are all different. This is due in part to a variation in choice of W and of its dimensions d, and in part to the use of alternative ordering relations in V. Thus in some schemes it is by no means obvious that each simple root given is positive. One advantage of the lexicographic ordering in W used throughout this paper is that it is trivial to determine whether or not a particular root is positive or negative. However, the main point of distinction between rival labelling schemes concerns the choice of W and d.

As pointed out by Dynkin (1957b, p 353) and stressed by Wybourne and Bowick (1977), in dealing with an exceptional simple Lie algebra g of rank k, it is convenient to make use of certain maximal embeddings in g of a classical semi-simple Lie algebra h of the same rank k. The relevant embeddings may be referred to as natural. They are necessarily regular in the sense defined by Dynkin (1957a, p 142) and may be obtained from those listed in his table 12. To be precise, the natural embeddings of a classical semi-simple Lie algebra h in an exceptional simple Lie algebra g are those that satisfy three criteria: firstly, the Cartan subalgebras of g and h are of the same dimension, with the Cartan subalgebra of h a subalgebra of the Cartan subalgebra of g; secondly, the root vectors $\boldsymbol{r}$ of h are root vectors of g as they stand, without any change of basis or change of scale in the corresponding space W containing the root spaces of both g and h; thirdly, the embedding is classically maximal in the sense that there exists no classical semi-simple Lie algebra h^* such that $g \supset h^* \supset h$ with h^* not isomorphic to h.

The required natural embeddings may be found by the following the prescription, described by Dynkin (1957a, p 145), based on the extension of the simple root system Π_g by the adjunction of the lowest root of g, which is necessarily negative, to give an extended Dynkin diagram. These are exhibited in table 1. The prescription involves the deletion of one of the simple roots of the original Dynkin diagram from the extended diagram. The resulting diagram is the Dynkin diagram of a regular semi-simple subalgebra of g. The results obtained in this way are displayed in Dynkin's table 12. Of these embeddings in the simple exceptional Lie algebras, the following involve a change of scale at variance with the second criterion mentioned above:

$$G_2 \supset A_1 \dotplus A_1, \tag{2.2}$$

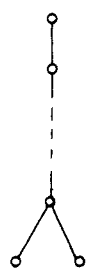
$$F_4 \supset A_2 \dotplus A_2, \tag{2.3}$$

$$F_4 \supset A_1 \dotplus A_3. \tag{2.4}$$

The third criterion of classical maximality is such that consideration has to be given to repeating Dynkin's procedure until a solely classical semi-simple Lie algebra is reached.

Contrary to the claim made by Dynkin, not all the embeddings listed in his table 12 are maximal, since the possibility of embedding one regular subalgebra listed in another listed regular subalgebra seems to have been overlooked. This matter is taken up in the

Table 2.

Group G	Algebra g	Complementary system $\boldsymbol{p} \in \Gamma_g$	Simple root system of $g : \boldsymbol{r} \in \Pi_g$
SU$(k+1)$	A_k	$\boldsymbol{e}_1+\boldsymbol{e}_2+\ldots+\boldsymbol{e}_{k+1}$	 $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_2-\boldsymbol{e}_3$, …, $\boldsymbol{e}_{k-2}-\boldsymbol{e}_{k-1}$, $\boldsymbol{e}_{k-1}-\boldsymbol{e}_k$, $\boldsymbol{e}_k-\boldsymbol{e}_{k+1}$
SO$(2k+1)$	B_k	—	 $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_2-\boldsymbol{e}_3$, …, $\boldsymbol{e}_{k-2}-\boldsymbol{e}_{k-1}$, $\boldsymbol{e}_{k-1}-\boldsymbol{e}_k$, $\boldsymbol{e}_k$
Sp$(2k)$	C_k	—	 $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_2-\boldsymbol{e}_3$, …, $\boldsymbol{e}_{k-2}-\boldsymbol{e}_{k-1}$, $\boldsymbol{e}_{k-1}-\boldsymbol{e}_k$, $2\boldsymbol{e}_k$
SO$(2k)$	D_k	—	 $\boldsymbol{e}_1-\boldsymbol{e}_2$, $\boldsymbol{e}_2-\boldsymbol{e}_3$, …, $\boldsymbol{e}_{k-2}-\boldsymbol{e}_{k-1}$, $\boldsymbol{e}_{k-1}+\boldsymbol{e}_k$, $\boldsymbol{e}_{k-1}-\boldsymbol{e}_k$

next section. It suffices to say at this stage that the complete list of natural embeddings of a classical semi-simple Lie algebra h in an exceptional simple Lie algebra g is given in table 3.

The root systems corresponding to these embeddings may be established immediately, by making use of the branching rules for the adjoint or regular representations of g into irreducible representations of h. These may be found, for example, in the work of Patera and Sankoff (1972) and of Wybourne and Bowick (1977). The relevant results

Table 3.

Exceptional algebra g	Classical subalgebra h	Branching of regular representation of g	h-dominant roots of g
G_2	A_2	$(21) \to \{21\}$ $\{1\}$ $\{1^2\}$	$\bar{1}01$ $\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}$ $\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}$
F_4	B_4	$(1^2) \to [1^2]$ Δ	$1100\ \ 1000$ $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$
F_4	$C_1 \dot{+} C_3$	$(2) \to \langle 2\rangle\langle 0\rangle$ $\langle 1\rangle\langle 1^3\rangle$ $\langle 0\rangle\langle 2\rangle$	$2:000$ $1:111\ \ 1:100$ $0:200\ \ 0:110$
E_6	$A_1 \dot{+} A_5$	$(2) \to \{2\}\{0\}$ $\{1\}\{1^3\}$ $\{0\}\{21^4\}$	$1\bar{1}:000000$ $\frac{1}{2}\frac{\bar{1}}{2}:\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}$ $00:10000\bar{1}$
E_6	$A_2 \dot{+} A_2 \dot{+} A_2$	$(21) \to \{21\}\{0\}\{0\}$ $\{1\}\{1^2\}\{1\}$ $\{1^2\}\{1\}\{1^2\}$ $\{0\}\{21\}\{0\}$ $\{0\}\{0\}\{21\}$	$10\bar{1}:000:000$ $\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}:\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}$ $\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}:\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}$ $000:101:000$ $000:000:101$
E_7	A_7	$(21^6) \to \{21^6\}$ $\{1^4\}$	$1000000\bar{1}$ $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}$
E_7	$A_1 \dot{+} D_6$	$(2) \to \{2\}[0]$ $\{1\}\Delta_-$ $\{0\}[1^2]$	$1\bar{1}:000000$ $\frac{1}{2}\frac{\bar{1}}{2}:\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{\bar{1}}{2}$ $00:110000$
E_7	$A_2 \dot{+} A_5$	$(21) \to \{21\}\{0\}$ $\{1\}\{1^4\}$ $\{1^2\}\{1^2\}$ $\{0\}\{21^4\}$	$10\bar{1}:000000$ $\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}:\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}\frac{\bar{2}}{3}$ $\frac{1}{3}\frac{1}{3}\frac{2}{3}:\frac{2}{3}\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}$ $000:10000\bar{1}$
E_8	A_8	$(21^7) \to \{21^7\}$ $\{1^3\}$ $\{1^6\}$	$10000000\bar{1}$ $\frac{2}{3}\frac{2}{3}\frac{2}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}\frac{\bar{1}}{3}$ $\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{\bar{2}}{3}\frac{\bar{2}}{3}\frac{\bar{2}}{3}$
E_8	D_8	$(1^2) \to [1^2]$ Δ_+	11000000 $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$
E_8	$A_7 \dot{+} A_1$	$(21^6) \to \{21^6\}\{0\}$ $\{1^2\}\{1\}$ $\{1^4\}\{0\}$ $\{1^6\}\{1\}$ $\{0\}\{2\}$	$1000000\bar{1}:00$ $\frac{3}{4}\frac{3}{4}\frac{\bar{1}}{4}\frac{\bar{1}}{4}\frac{\bar{1}}{4}\frac{\bar{1}}{4}\frac{\bar{1}}{4}\frac{\bar{1}}{4}:\frac{1}{2}\frac{\bar{1}}{2}$ $\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}\frac{\bar{1}}{2}:00$ $\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{\bar{3}}{4}\frac{\bar{3}}{4}:\frac{1}{2}\frac{\bar{1}}{2}$ $00000000:1\bar{1}$

Table 3.—continued

Exceptional algebra g	Classical subalgebra h	Branching of regular representation of g	h-dominant roots of g
E_8	$A_4 \dotplus A_4$	$(21^3) \to \{21^3\}\{0\}$	$10001:00000$
		$\{1\}\{1^3\}$	$\frac{4}{5}\frac{\bar1}{5}\frac{\bar1}{5}\frac{\bar1}{5}\frac{\bar1}{5}:\frac{2}{5}\frac{2}{5}\frac{2}{5}\frac{\bar3}{5}\frac{\bar3}{5}$
		$\{1^2\}\{1^4\}$	$\frac{3}{5}\frac{3}{5}\frac{\bar2}{5}\frac{\bar2}{5}\frac{\bar2}{5}:\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{\bar4}{5}$
		$\{1^3\}\{1\}$	$\frac{2}{5}\frac{2}{5}\frac{2}{5}\frac{\bar3}{5}\frac{\bar3}{5}:\frac{4}{5}\frac{\bar1}{5}\frac{\bar1}{5}\frac{\bar1}{5}\frac{\bar1}{5}$
		$\{1^4\}\{1^2\}$	$\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{1}{5}\frac{\bar4}{5}:\frac{3}{5}\frac{3}{5}\frac{\bar2}{5}\frac{\bar2}{5}\frac{\bar2}{5}$
		$\{0\}\{21^3\}$	$00000:1000\bar1$
E_8	$A_5 \dotplus A_2 \dotplus A_1$	$(21^4) \to \{21^4\}\{0\}\{0\}$	$10000\bar1:000:00$
		$\{1\}\{1\}\{1\}$	$\frac{5}{6}\frac{\bar1}{6}\frac{\bar1}{6}\frac{\bar1}{6}\frac{\bar1}{6}\frac{\bar1}{6}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{1}{2}\frac{\bar1}{2}$
		$\{1^2\}\{1^2\}\{0\}$	$\frac{2}{3}\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:00$
		$\{1^3\}\{0\}\{1\}$	$\frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{\bar1}{2}\frac{\bar1}{2}\frac{\bar1}{2}:000:\frac{1}{2}\frac{\bar1}{2}$
		$\{1^4\}\{1\}\{0\}$	$\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}\frac{\bar2}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:00$
		$\{1^5\}\{1^2\}\{1\}$	$\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{1}{6}\frac{\bar5}{6}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{1}{2}\frac{\bar1}{2}$
		$\{0\}\{21\}\{0\}$	$000000:10\bar1:00$
		$\{0\}\{0\}\{2\}$	$000000:000:1\bar1$
E_8	$A_2 \dotplus A_2 \dotplus A_2 \dotplus A_2$	$(21) \to \{21\}\{0\}\{0\}\{0\}$	$10\bar1:000:000:000$
		$\{1\}\{1\}\{0\}\{1^2\}$	$\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:000:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}$
		$\{1\}\{1^2\}\{1^2\}\{0\}$	$\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:000$
		$\{1\}\{0\}\{1\}\{1\}$	$\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:000:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}$
		$\{1^2\}\{1^2\}\{0\}\{1\}$	$\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:000:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}$
		$\{1^2\}\{1\}\{1\}\{0\}$	$\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:000$
		$\{1^2\}\{0\}\{1^2\}\{1^2\}$	$\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:000:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}$
		$\{0\}\{21\}\{0\}\{0\}$	$000:10\bar1:000:000$
		$\{0\}\{1\}\{1^2\}\{1\}$	$000:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}$
		$\{0\}\{1^2\}\{1\}\{1^2\}$	$000:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}:\frac{2}{3}\frac{\bar1}{3}\frac{\bar1}{3}:\frac{1}{3}\frac{1}{3}\frac{\bar2}{3}$
		$\{0\}\{0\}\{21\}\{0\}$	$000:000:10\bar1:000$
		$\{0\}\{0\}\{0\}\{21\}$	$000:000:000:10\bar1$

are displayed in table 3 in a notation which will be explained fully in § 4. The complete root system Σ_g is then given by the non-zero weights of the irreducible representations of the classical Lie algebras which appear in these branchings. These may be obtained, for example, from the work of King and Plunkett (1976). It is important to specify the weight vectors of the irreducible representations of the simple constituents h_1, h_2, of the classical semi-simple algebra $h = h_1 \dotplus h_2 \dotplus \ldots$ as vectors in the root spaces V_1, $V_2, \ldots$ of these constituents with components defined with respect to the basis vectors of the corresponding Euclidean spaces W_1, $W_2, \ldots$. This has been done for all the non-zero weights of the irreducible representations of h, which are dominant, that is to say are highest weights of some irreducible representation. These weights, appearing in the final column of table 3, are thus the h-dominant roots of the exceptional algebra g; the remaining roots may be obtained through the action of the Weyl symmetry group appropriate to each simple constituent h_i of h. This group permutes in all possible ways the components of vectors in W_i, and in the cases for which $h_i = B_j$ or C_j also changes the signs of these components in all possible ways, whilst if $h_i = Dj$ it changes the signs of pairs of components in all possible ways. This action generates the complete root system Σ_g.

The Euclidean space W associated with the root system of g is the direct sum of the corresponding spaces $W_1, W_2, \ldots$ of $h_1, h_2, \ldots$. The dimension d of the space W spanned by the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_d$ is thus the sum of the dimensions $d_1, d_2, \ldots$ of $W_1, W_2, \ldots$. The precise mode of embedding of $W_1, W_2, \ldots$ in W is to some extent arbitrary. The method adopted here for the case $g \supset h_1 \dotplus h_2 \dotplus \ldots$ results, in general, in an alteration of the ordering of the unit vectors implicit in the notation of table 3. It involves dealing with each simple algebra in turn, first h_1, then h_2, etc. The root space of each such algebra h of rank j is then embedded in a space spanned by $\boldsymbol{e}_i, \boldsymbol{e}_{i+1}, \ldots, \boldsymbol{e}_{i+j-1}$ and $\boldsymbol{e}_m$, where i is chosen as small as possible, and $m = i+j$ if $h \neq \mathrm{A}_j$ but m is as large as possible if $h = \mathrm{A}_j$. Each vector $\boldsymbol{p}$ spanning $V^{\perp}$ is associated with an algebra $h = \mathrm{A}_j$ and takes the form

$$\boldsymbol{p} = \boldsymbol{e}_i + \boldsymbol{e}_{i+1} + \ldots + \boldsymbol{e}_{i+j-1} + \boldsymbol{e}_m. \tag{2.5}$$

This particular method of constructing W as the disjoint union of W_1, W_2, etc has certain advantages that become apparent when discussing irreducible representations.

From the definition of a simple root, it follows that the simple root system, Π_g, may be recovered from the complete root system, Σ_g, merely by eliminating from the set of positive roots all those which appear in the set of all possible sums of two such roots. However, an easier algorithm presents itself which is based on the fact that the simple roots span V. The highest simple root is the lowest root vector with a positive first component, the next highest is the lowest root vector with zero first component but positive second component, etc. This procedure for extracting the simple roots from the set of all roots is completed after precisely k steps in the case of most rank-k Lie algebras. The only exceptions are associated with the classical simple algebra D_k. If g is D_k itself, then to complete the set of simple roots, Π_g, it is necessary to include the root $\boldsymbol{e}_{k-1} + \boldsymbol{e}_k$, whilst if g is exceptional and contains D_j as a subalgebra for some j, it may be necessary to complete the set of roots by including the analogue of the above root, namely $\boldsymbol{e}_{i+j-1} + \boldsymbol{e}_{i+j}$.

Applying this technique to the root systems of the exceptional simple Lie algebras, g, defined by the weights of the representations of the classical semi-simple natural Lie subalgebras h of table 3, yields the simple root systems of g diplayed in table 4 in association with the corresponding Dynkin diagrams.

The simple roots $\boldsymbol{s}$ of g which are not roots, simple or otherwise, of h stand out as being those of quite a distinct form. The corresponding roots $\boldsymbol{q}$ of g which are not simple roots of g but are simple roots of h are also indicated. They are joined to the Dynkin diagram of g by means of broken lines and in this way yield supplemented diagrams. These look similar to, but should not be confused with, the extended diagrams of Dynkin (1957a, p 145) which were discussed earlier.

In one case exhibited in table 4 a doubly supplemented diagram appears. A comparison of the simple root system of E_8 obtained in this case with $h = \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2$, and the simple root system of E_6 obtained in the case $h = \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2$, shows that this double supplementation owes its origin to the existence of the subalgebra chain

$$\mathrm{E}_8 \supset \mathrm{A}_2 \dotplus \mathrm{E}_6 \supset \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2. \tag{2.6}$$

This merely corresponds to the fact that the embedding of $\mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2 \dotplus \mathrm{A}_2$ in E_8 is classically maximal, as required here, but not maximal.

Table 4.

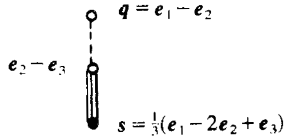
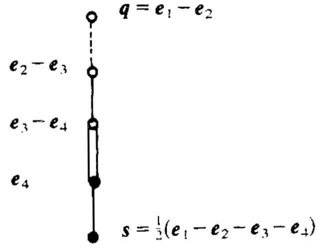
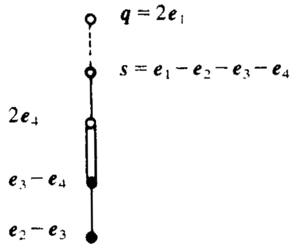
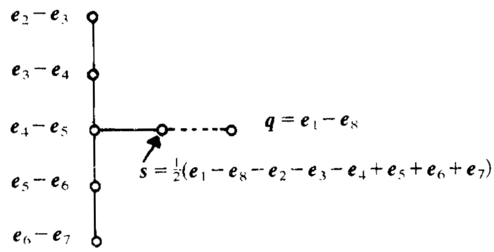
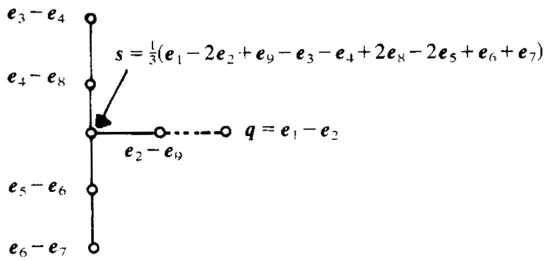
Exceptional algebra g	Classical subalgebra h	Complementary system $p \in \Gamma_g$	Simple roots systems of g and $h: s \in \Pi_g$, $q \in \Pi_h$
G_2	A_2	$e_1 + e_2 + e_3$	
F_4	B_4	—	
F_4	$C_1 \dot{+} C_3$	—	
E_6	$A_1 \dot{+} A_5$	$e_1 + e_8$ $e_2 + e_3 + \ldots + e_7$	
E_6	$A_2 \dot{+} A_2 \dot{+} A_2$	$e_1 + e_2 + e_9$ $e_3 + e_4 + e_8$ $e_5 + e_6 + e_7$	

Table 4 —continued

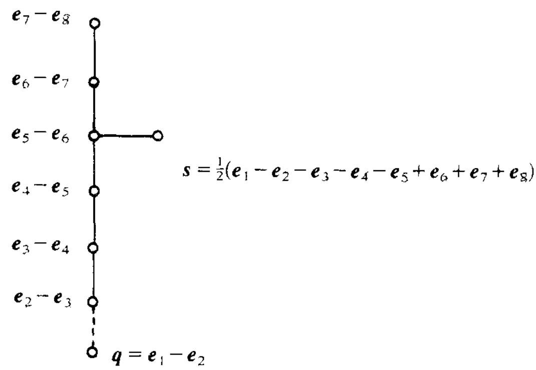
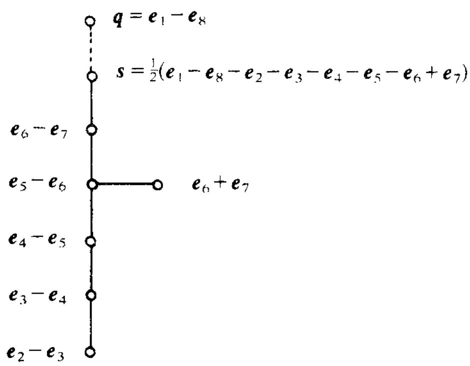
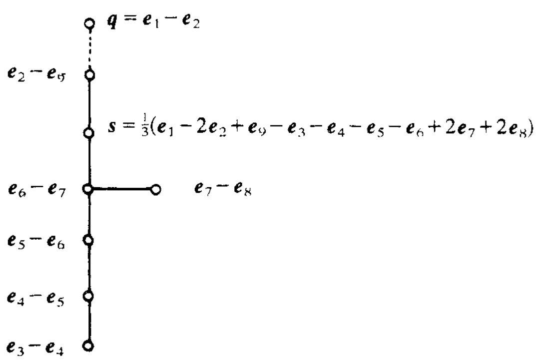
<table>
<tr><th>Exceptional algebra g</th><th>Classical subalgebra h</th><th>Complementary system $p \in \Gamma_g$</th><th>Simple root systems of g and $h: s \in \Pi_g$, $q \in \Pi_h$</th></tr>
<tr><td>E_7</td><td>A_7</td><td>$e_1+e_2+\ldots+e_8$</td><td>
</td></tr>
<tr><td>E_7</td><td>$A_1 \dot{+} D_6$</td><td>e_1+e_8</td><td>
</td></tr>
<tr><td>E_7</td><td>$A_2 \dot{+} A_5$</td><td>$e_1+e_2+e_9$
$e_3+e_4+\ldots+e_8$</td><td>
</td></tr>
</table>

Table 4 —continued

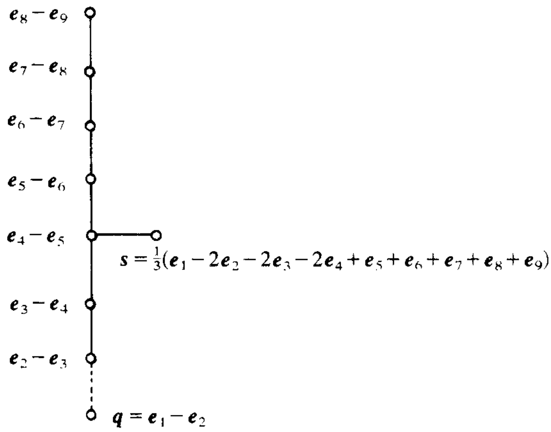
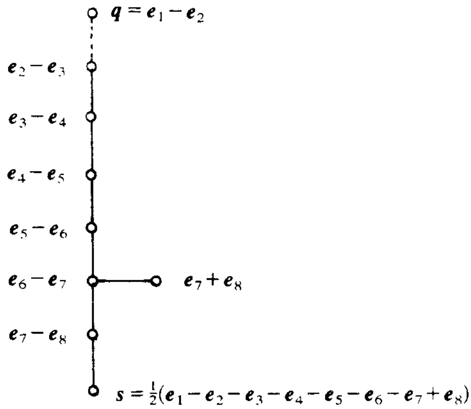
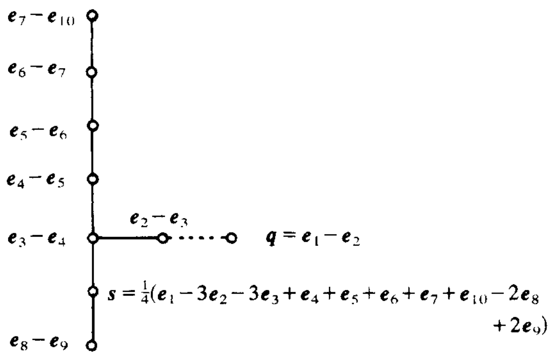
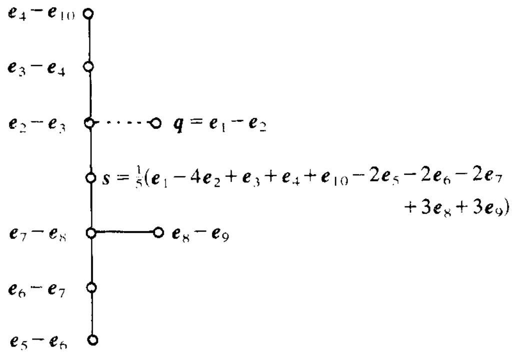
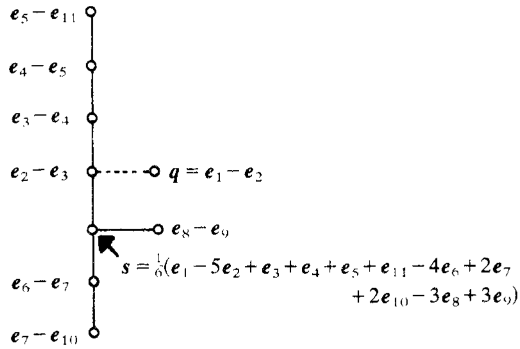
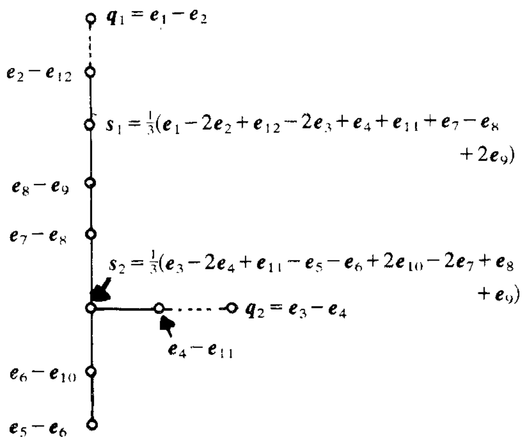
Exceptional algebra g	Classical subalgebra h	Complementary system $p \in \Gamma_g$	Simple root systems of g and $h: s \in \Pi_g$, $q \in \Pi_h$
E_8	A_8	$e_1+e_2+\ldots+e_9$	
E_8	D_8	—	
E_8	$A_7\dot{+}A_1$	$e_1+e_2+\ldots$ $\ldots+e_7+e_{10}$ e_8+e_9	

Table 4—continued

Exceptional algebra g	Classical subalgebra h	Complementary system $p \in \Gamma_g$	Simple root systems of g and $h: s \in \Pi_g$, $q \in \Pi_h$
E_8	$A_4 \dot{+} A_4$	$e_1+e_2+e_3+e_4+e_{10}$ $e_5+e_6+e_7+e_8+e_9$	
E_8	$A_5 \dot{+} A_2 \dot{+} A_1$	$e_1+e_2+e_3+e_4+e_5+e_{11}$ $e_6+e_7+e_{10}$ e_8+e_9	
E_8	$A_2 \dot{+} A_2 \dot{+} A_2 \dot{+} A_2$	$e_1+e_2+e_{12}$ $e_3+e_4+e_{11}$ $e_5+e_6+e_{10}$ $e_7+e_8+e_9$	

3. Chains of regular embeddings

There are some apparently surprising omissions from the list of natural embeddings given in table 3. This is related to the fact that not all embeddings associated with singly supplemented Dynkin diagrams are necessarily maximal. To see this, it is instructive to consider what seems a rather innocuous alteration in the work of the previous section: namely a reordering of the simple constituents $h_1, h_2, \ldots$ of the classical semi-simple Lie subalgebra $h = h_1 \dotplus h_2 \dotplus \ldots$ of the exceptional simple Lie algebra g.

This reordering yields, using once more the weights of the representations of the classical subalgebras of table 3, the simple root systems and supplemented Dynkin diagrams of table 5.

In the case of F_4 this alteration does not affect the simple root system at all. However, the corresponding supplemented diagram is quite distinct from the original one—it is doubly supplemented with the simple roots $\boldsymbol{e}_1 - \boldsymbol{e}_2$ and $2\boldsymbol{e}_3$ of C_3. It might appear from this diagram that the embedding of $C_3 \dotplus C_1$ in F_4 may not be maximal. However, it is maximal since the algebra C_4, which the diagram correctly implies contains $C_3 \dotplus C_1$ as a subgroup, is not itself a subgroup of F_4. This may easily be verified by looking at the complete root system of F_4 and the known weights of the adjoint and other irreducible representations of C_4.

For E_6 nothing unusual emerges, in the sense that the simple root systems obtained using $E_8 \supset A_1 \dotplus A_5$ and $E_6 \supset A_5 \dotplus A_1$ are essentially the same, that is, related to one another by a permutation of the basis vectors $\boldsymbol{e}_i$ of W. The same is true for E_7. However, the simple root system of E_8 obtained using $E_8 \supset A_1 \dotplus A_7$ is markedly different from that obtained using $E_8 \supset A_7 \dotplus A_1$. Moreover, the corresponding supplemented Dynkin diagram is doubly supplemented rather than singly supplemented, even though the subalgebra is the same in both cases. What is most significant, however, is that the double supplementation necessary to yield the simple root system of $A_1 \dotplus A_7$ proceeds via a supplementation yielding what is clearly the simple root system of $A_1 \dotplus E_7$, a known subgroup of E_8, followed by another supplementation of the simple root system of E_7 to give that of A_7. Examination of the comlete root system of E_8, as defined by the branching rules for the adjoint representation, shows that the embedding of $A_1 \dotplus A_7$ in E_8 is not maximal as claimed by Dynkin (1957a, p 150) and Patera and Sankoff (1972). The embedding is associated with the subalgebra chain

$$E_8 \supset A_1 \dotplus E_7 \supset A_1 \dotplus A_7. \tag{3.1}$$

Similarly, alterations of the subalgebra $A_5 \dotplus A_2 \dotplus A_1$ of E_8 to $A_1 \dotplus A_2 \dotplus A_5$ and $A_2 \dotplus A_1 \dotplus A_5$ give the simple root systems of table 5 which show immediately the existence of the chains

$$E_8 \supset A_1 \dotplus E_7 \supset A_1 \dotplus A_2 \dotplus A_5 \tag{3.2}$$

and

$$E_8 \supset A_2 \dotplus E_6 \supset A_2 \dotplus A_1 \dotplus A_5. \tag{3.3}$$

Consideration of another three permutations is obviously possible but results in nothing new.

Now it is perhaps clear why the list of natural embeddings given in table 3 is so short. All the remaining embeddings expected are neither maximal nor even classically

Table 5.

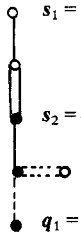
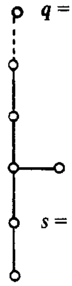
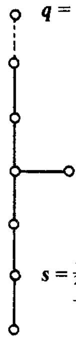
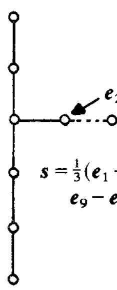
Exceptional Lie algebra g	Classical Lie subalgebra h	Complementary system $p \in \Gamma_g$		Simple root systems of g and $H: s \in \Pi_g$, $q \in \Pi_h$
F_4	$C_3 \dotplus C_1$	—	$2e_4$ $e_2 - e_3$	 $s_1 = e_1 - e_2 - e_3 - e_4$ $s_2 = e_3 - e_4$ $q_2 = 2e_3$ $q_1 = e_1 - e_2$
E_6	$A_5 \dotplus A_1$	$e_1 + e_2 + \ldots + e_5 + e_8$ $e_6 + e_7$	$e_2 - e_3$ $e_3 - e_4$ $e_4 - e_5$ $e_6 - e_7$	 $q = e_1 - e_2$ $e_5 - e_8$ $s = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 + e_5 + e_8 - e_6 + e_7)$
E_7	$D_6 \dotplus A_1$	$e_7 + e_8$	$e_2 - e_3$ $e_3 - e_4$ $e_4 - e_5$ $e_5 - e_6$ $e_7 - e_8$	 $q = e_1 - e_2$ $e_5 + e_6$ $s = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 + e_6 - e_7 + e_8)$
E_7	$A_5 \dotplus A_2$	$e_1 + e_2 + \ldots + e_5 + e_9$ $e_6 + e_7 + e_8$	$e_5 - e_9$ $e_4 - e_5$ $e_3 - e_4$ $e_7 - e_8$ $e_6 - e_7$	 $e_2 - e_3$ $q = e_1 - e_2$ $s = \frac{1}{3}(e_1 - 2e_2 - 2e_3 + e_4 + e_5 + e_9 - e_6 - e_7 + 2e_8)$

Table 5—continued

Exceptional Lie algebra g	Classical Lie subalgebra h	Complementary system $\boldsymbol{p}\in\Gamma_g$		Simple root systems of g and $H: \boldsymbol{s}\in\Pi_g$, $\boldsymbol{q}\in\Pi_h$
				$\boldsymbol{q}_1=\boldsymbol{e}_1-\boldsymbol{e}_{10}$
E_8	$A_1\dot{+}A_7$	$\boldsymbol{e}_1+\boldsymbol{e}_{10}$		$\boldsymbol{s}=\frac{1}{4}(2\boldsymbol{e}_1-2\boldsymbol{e}_{10}-3\boldsymbol{e}_2-3\boldsymbol{e}_3+\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9)$
		$\boldsymbol{e}_2+\boldsymbol{e}_3+\ldots+\boldsymbol{e}_9$	$\boldsymbol{e}_3-\boldsymbol{e}_4$	$\boldsymbol{q}_2=\boldsymbol{e}_2-\boldsymbol{e}_3$
			$\boldsymbol{e}_4-\boldsymbol{e}_5$	
			$\boldsymbol{e}_5-\boldsymbol{e}_6$	
			$\boldsymbol{e}_6-\boldsymbol{e}_7$	$\boldsymbol{s}_2=\frac{1}{2}(\boldsymbol{e}_2-\boldsymbol{e}_3-\boldsymbol{e}_4-\boldsymbol{e}_5-\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9)$
			$\boldsymbol{e}_7-\boldsymbol{e}_8$	
			$\boldsymbol{e}_8-\boldsymbol{e}_9$	
				$\boldsymbol{q}_1=\boldsymbol{e}_1-\boldsymbol{e}_{11}$
E_8	$A_1\dot{+}A_2\dot{+}A_5$	$\boldsymbol{e}_1+\boldsymbol{e}_{11}$		$\boldsymbol{s}_1=\frac{1}{6}(3\boldsymbol{e}_1-3\boldsymbol{e}_{11}-4\boldsymbol{e}_2+2\boldsymbol{e}_3+2\boldsymbol{e}_{10}-5\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9)$
		$\boldsymbol{e}_2+\boldsymbol{e}_3+\boldsymbol{e}_{10}$	$\boldsymbol{e}_4-\boldsymbol{e}_5$	
		$\boldsymbol{e}_4+\boldsymbol{e}_5+\ldots+\boldsymbol{e}_9$	$\boldsymbol{e}_5-\boldsymbol{e}_6$	
			$\boldsymbol{e}_6-\boldsymbol{e}_7$	
			$\boldsymbol{e}_7-\boldsymbol{e}_8$	$\boldsymbol{e}_8-\boldsymbol{e}_9$
				$\boldsymbol{s}_2=\frac{1}{3}(\boldsymbol{e}_2-2\boldsymbol{e}_3+\boldsymbol{e}_{10}-\boldsymbol{e}_4-\boldsymbol{e}_5-\boldsymbol{e}_6-\boldsymbol{e}_7+2\boldsymbol{e}_8+2\boldsymbol{e}_9)$
			$\boldsymbol{e}_3-\boldsymbol{e}_{10}$	
				$\boldsymbol{q}_2=\boldsymbol{e}_2-\boldsymbol{e}_3$
				$\boldsymbol{q}_1=\boldsymbol{e}_1-\boldsymbol{e}_2$
E_8	$A_2\dot{+}A_1\dot{+}A_5$	$\boldsymbol{e}_1+\boldsymbol{e}_2+\boldsymbol{e}_{11}$	$\boldsymbol{e}_2-\boldsymbol{e}_{11}$	
		$\boldsymbol{e}_3+\boldsymbol{e}_{10}$		$\boldsymbol{s}_1=\frac{1}{6}(2\boldsymbol{e}_1-4\boldsymbol{e}_2+2\boldsymbol{e}_{11}-3\boldsymbol{e}_3+3\boldsymbol{e}_{10}-5\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9)$
		$\boldsymbol{e}_4+\boldsymbol{e}_5+\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9$	$\boldsymbol{e}_4-\boldsymbol{e}_5$	
			$\boldsymbol{e}_5-\boldsymbol{e}_6$	
			$\boldsymbol{e}_6-\boldsymbol{e}_7$	$\boldsymbol{q}_2=\boldsymbol{e}_3-\boldsymbol{e}_{10}$
			$\boldsymbol{e}_7-\boldsymbol{e}_8$	$\boldsymbol{s}_2=\frac{1}{2}(\boldsymbol{e}_3-\boldsymbol{e}_{10}-\boldsymbol{e}_4-\boldsymbol{e}_5-\boldsymbol{e}_6+\boldsymbol{e}_7+\boldsymbol{e}_8+\boldsymbol{e}_9)$
			$\boldsymbol{e}_8-\boldsymbol{e}_9$	

maximal. They are associated with the chains

$$E_7 \supset A_1 \dotplus D_6 \supset A_1 \dotplus A_3 \dotplus A_3, \tag{3.4}$$

$$E_8 \supset A_1 \dotplus E_7 \supset A_1 \dotplus A_1 \dotplus D_6 \supset A_1 \dotplus A_1 \dotplus A_3 \dotplus A_3, \tag{3.5}$$

$$E_8 \supset D_8 \supset A_1 \dotplus A_1 \dotplus D_6, \tag{3.6}$$

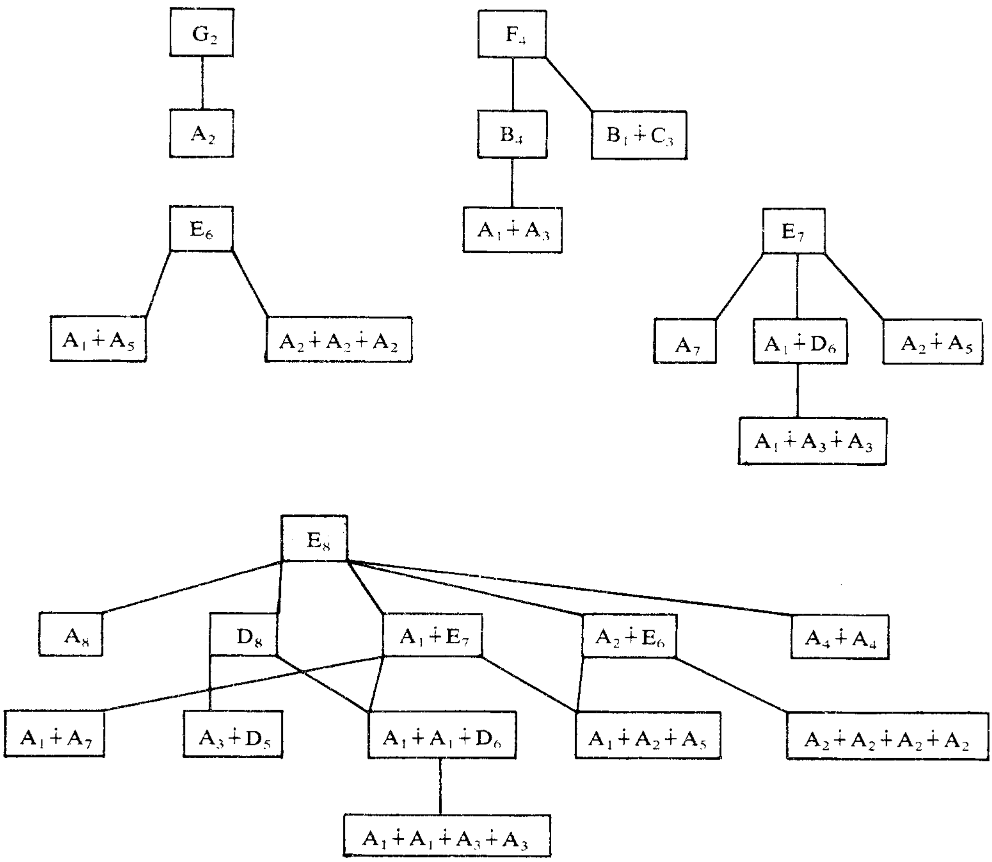
$$E_8 \supset D_8 \supset A_3 \dotplus D_5. \tag{3.7}$$

For completeness, it should be pointed out that the embedding (2.4), discarded already on the grounds that it involves a change of scale, is also not maximal in the sense that it is associated with the chain

$$F_4 \supset B_4 \supset A_1 \dotplus A_3. \tag{3.8}$$

These embeddings and those referred to earlier are summarised in table 6. The

Table 6. Regular subalgebra chains.



corresponding root systems of E_7 and E_8 are defined by the weights of the irreducible representations of the classical semi-simple subalgebras appearing in the branching of the regular representations of E_7 and E_8. These branching rules are given in table 7, and labelling schemes for the systems of simple roots are displayed in table 8. It is the supplemented Dynkin diagrams of this table which make manifest the subalgebra chains. Once again it is clear that the ordering of the simple constituents of the classical subalgebra $h_1 \dotplus h_2 \dotplus \dots$ determine whether or not a doubly supplemented diagram is obtained which illustrates the non-maximal nature of the embedding. It should be stressed that the existence of such a diagram does not prove the non-maximality. This has to be demonstrated by examining the complete root system via the branching rules given in tables 3 and 7 and others appearing in the tabulations of Patera and Sankoff (1972) and of Wybourne and Bowick (1977). Finally, due credit should be given to Wybourne (1979) who was the first to point out and indeed use the existence and interconnection between the chains (3.2) and (3.3).

Table 7.

Exceptional algebra g	Classical subalgebra h	Branching of regular representation of g	h-dominant roots of g
E_7	$A_1 \dotplus A_3 \dotplus A_3$	$(2) \to \{2\}\{0\}\{0\}$	$1\bar{1}:0000:0000$
		$\{1\}\{1\}\{1^3\}$	$\tfrac{\bar{1}}{2}\tfrac{1}{2}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}:\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{\bar{3}}{4}$
		$\{1\}\{1^3\}\{1\}$	$\tfrac{1}{2}\tfrac{1}{2}:\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{\bar{3}}{4}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}$
		$\{0\}\{21\}\{0\}$	$00:100\bar{1}:0000$
		$\{0\}\{1^2\}\{1^2\}$	$00:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}$
		$\{0\}\{0\}\{21^2\}$	$00:0000:100\bar{1}$
E_8	$D_5 \dotplus A_3$	$(1^2) \to [1^2]\{0\}$	$11000:0000$
		$[1]\{1^2\}$	$10000:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}$
		$\Delta_+\{1\}$	$\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}$
		$\Delta_-\{1^3\}$	$\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{\bar{3}}{4}$
		$[0]\{21^2\}$	$00000:100\bar{1}$
E_8	$D_6 \dotplus A_1 \dotplus A_1$	$(1^2) \to [1^2]\{0\}\{0\}$	$110000:00:00$
		$[1]\{1\}\{1\}$	$100000:\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{\bar{1}}{2}$
		$\Delta_+\{0\}\{1\}$	$\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}:00:\tfrac{1}{2}\tfrac{\bar{1}}{2}$
		$\Delta_-\{1\}\{0\}$	$\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{\bar{1}}{2}:00$
		$[0]\{2\}\{0\}$	$000000:1\bar{1}:00$
		$[0]\{0\}\{2\}$	$000000:00:1\bar{1}$
E_8	$A_1 \dotplus A_1 \dotplus A_3 \dotplus A_3$	$(2) \to \{2\}\{0\}\{0\}\{0\}$	$1\bar{1}:00:0000:0000$
		$\{1\}\{1\}\{1^2\}\{0\}$	$\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}:0000$
		$\{1\}\{1\}\{0\}\{1^2\}$	$\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{\bar{1}}{2}:0000:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}$
		$\{1\}\{0\}\{1\}\{1\}$	$\tfrac{1}{2}\tfrac{\bar{1}}{2}:00:\tfrac{3}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}$
		$\{1\}\{0\}\{1^2\}\{1^2\}$	$\tfrac{1}{2}\tfrac{\bar{1}}{2}:00:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}$
		$\{0\}\{2\}\{0\}\{0\}$	$00:1\bar{1}:0000:0000$
		$\{0\}\{1\}\{1\}\{1^3\}$	$00:\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}:\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{\bar{3}}{4}$
		$\{0\}\{1\}\{1^3\}\{1\}$	$00:\tfrac{1}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{4}\tfrac{1}{4}\tfrac{1}{4}\tfrac{\bar{3}}{4}:\tfrac{3}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}\tfrac{\bar{1}}{4}$
		$\{0\}\{0\}\{21^2\}\{0\}$	$00:00:100\bar{1}:0000$
		$\{0\}\{0\}\{1^2\}\{1^2\}$	$00:00:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}:\tfrac{1}{2}\tfrac{1}{2}\tfrac{\bar{1}}{2}\tfrac{\bar{1}}{2}$
		$\{0\}\{0\}\{0\}\{21^2\}$	$00:00:0000:100\bar{1}$

4. Labelling irreducible representations

Dynkin (1957b, p 329) has demonstrated that each irreducible representation $\boldsymbol{\lambda}$ of a semi-simple Lie algebra g and the corresponding Lie group G may be labelled by means of a vector $\boldsymbol{a}$ of non-negative integers with components a_α associated with each simple root $\boldsymbol{r}(\alpha)$ of the algebra. Each such component a_α, with $\alpha = 1, 2, \ldots, k$ where k is the

Table 8.

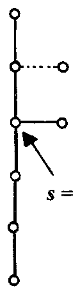
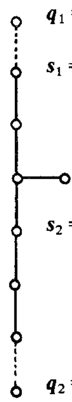
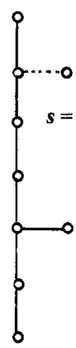
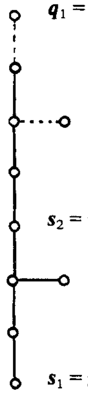
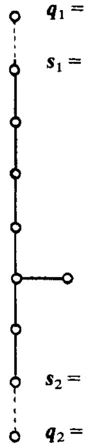
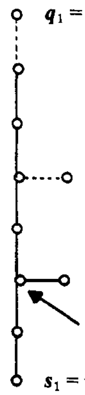
Exceptional algebra g	Classical subalgebra h	Complementary system $\boldsymbol{p} \in \Gamma_g$	Simple root systems of g and $h : \boldsymbol{s} \in \Pi_g$, $\boldsymbol{q} \in \Pi_h$
E_7	$A_3 \dotplus A_3 \dotplus A_1$	$e_1+e_2+e_3+e_{10}$ $e_4+e_5+e_6+e_9$ e_7+e_8	e_3-e_{10} e_2-e_3 $q = e_1-e_2$ e_7-e_8 $s = \frac{1}{4}(e_1-3e_2+e_3+e_{10}-e_4-e_5-e_6+3e_9-2e_7+2e_8)$ e_6-e_9 e_5-e_6 e_4-e_5 
E_7	$A_1 \dotplus A_3 \dotplus A_3$	e_1+e_{10} $e_2+e_3+e_4+e_9$ $e_5+e_6+e_7+e_8$	$q_1 = e_1-e_{10}$ $s_1 = \frac{1}{4}(2e_1-2e_{10}-3e_2+e_3+e_4+e_9-e_5-e_6-e_7+3e_8)$ e_7-e_8 e_6-e_7 e_5-e_6 $s_2 = \frac{1}{2}(e_2-e_3-e_4+e_9-e_5-e_6+e_7+e_8)$ e_4-e_9 e_3-e_4 $q_2 = e_2-e_3$ 
E_8	$A_3 \dotplus D_5$	$e_1+e_2+e_3+e_9$	e_3-e_9 e_2-e_3 $q = e_1-e_2$ $s = \frac{1}{4}(e_1-3e_2+e_3+e_9-2e_4-2e_5-2e_6-2e_7-2e_8)$ e_7+e_8 e_6-e_7 e_7-e_8 e_5-e_6 e_4-e_5 

Table 8—continued

Exceptional algebra g	Classical subalgebra h	Complementary system $\boldsymbol{p} \in \Gamma_g$	Simple root systems of g and $h : \boldsymbol{s} \in \Pi_g$, $\boldsymbol{q} \in \Pi_h$
E_8	$D_5 \dotplus A_3$	$e_6+e_7+e_8+e_9$	 $q_1 = e_1 - e_2$ $e_2 - e_3$ $e_3 - e_4$ $q_2 = e_4 + e_5$ $e_4 - e_5$ $s_2 = \frac{1}{2}(2e_5 - e_6 - e_7 + e_8 + e_9)$ $e_7 - e_8$ $e_6 - e_7$ $e_8 - e_9$ $s_1 = \frac{1}{4}(2e_1 - 2e_2 - 2e_3 - 2e_4 - 2e_5 - e_6 - e_7 - e_8 + 3e_9)$
E_8	$A_1 \dotplus A_1 \dotplus D_6$	e_1+e_{10} e_2+e_9	 $q_1 = e_1 - e_2$ $s_1 = \frac{1}{2}(e_1 - e_{10} - e_2 + e_{11} - 2e_3)$ $e_3 - e_4$ $e_4 - e_5$ $e_5 - e_6$ $e_6 - e_7$ $e_7 + e_8$ $e_7 - e_8$ $s_2 = \frac{1}{2}(e_2 - e_{11} - e_3 - e_4 - e_5 - e_6 - e_7 + e_8)$ $q_2 = e_2 - e_{11}$
E_8	$D_6 \dotplus A_1 \dotplus A_1$	e_7+e_{10} e_8+e_9	 $q_1 = e_1 - e_2$ $e_2 - e_3$ $e_3 - e_4$ $e_4 - e_5$ $q_2 = e_5 + e_6$ $e_5 - e_6$ $e_7 - e_{10}$ $s_2 = \frac{1}{2}(2e_6 - e_7 + e_{10} - e_8 + e_9)$ $e_8 - e_9$ $s_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4 - e_5 - e_6 - e_8 + e_9)$

rank of g, is given by

$$a_\alpha = 2\frac{\boldsymbol{M}\,.\,\boldsymbol{r}(\alpha)}{\boldsymbol{r}(\alpha)\,.\,\boldsymbol{r}(\alpha)} \tag{4.1}$$

where $\boldsymbol{M}$ is the highest weight vector of the irreducible representation $\boldsymbol{\lambda}$ introduced by Cartan. By hypothesis, $\boldsymbol{\lambda}$ is irreducible, and Cartan provided that in such a case $\boldsymbol{M}$ is unique and non-degenerate and serves to label the irreducible representation.

It should be stressed that whilst the vector $\boldsymbol{a}$ lies in some k-dimensional space, the vectors $\boldsymbol{M}$ and $\boldsymbol{r}(\alpha)$ lie in the particular k-dimensional space V which is a subspace of the d-dimensional space W introduced in § 2. The sense in which $\boldsymbol{M}$ is highest is precisely that associated with the ordering of vectors in W as defined previously. The fact that $\boldsymbol{M}$ lies wholly within V and not in $V^\perp$ is made explicit by the requirement that

$$\boldsymbol{M}\,.\,\boldsymbol{p} = 0 \tag{4.2}$$

for all vectors $\boldsymbol{p}$ of $V^\perp$.

The equations (4.1) and (4.2) are such that $\boldsymbol{a}$ and $\boldsymbol{M}$ may be obtained from one another in a straightforward way using the Euclidean metric of W, the simple roots $\boldsymbol{r}$ of g and the complementary vectors $\boldsymbol{p}$ given in tables 2 and 4.

In order to recover an irreducible representation label, $\boldsymbol{\lambda}$, more closely related to those used very widely for the classical Lie group representations, it is only necessary to note that this is accomplished in the case of the algebra A_k by adding $-M_{k+1}$ to each component of $\boldsymbol{M}$. This gives a vector $\boldsymbol{\lambda}$ with no more than k non-vanishing components. Generalising this procedure:

$$\boldsymbol{\lambda} = \boldsymbol{M} - \sum_{p\in\Gamma_g} M_m \boldsymbol{p} \tag{4.3}$$

where, in the notation of the previous section:

$$p = \boldsymbol{e}_i + \boldsymbol{e}_{i+1} + \ldots + \boldsymbol{e}_{i+j-1} + \boldsymbol{e}_m$$

and

$$M_m = \boldsymbol{M}\,.\,\boldsymbol{e}_m.$$

This definition ensures that the last $(d-k)$ components of $\boldsymbol{\lambda}$ are automatically zero by virtue of the precise way in which W_1, W_2, ... are embedded in W.

The first k components of $\boldsymbol{\lambda}$ are either integers or half odd integers thanks to the relation, following from (4.1), (4.3) and (2.1),

$$a_\alpha = 2\frac{\boldsymbol{\lambda}\,.\,\boldsymbol{r}(\alpha)}{\boldsymbol{r}(\alpha)\,.\,\boldsymbol{r}(\alpha)} \tag{4.4}$$

which links the components of $\boldsymbol{\lambda}$ to the non-negative integers a_α.

This prescription leads unambiguously to the standard labels $\{\boldsymbol{\lambda}\}$, $[\boldsymbol{\lambda}]$, $\langle\boldsymbol{\lambda}\rangle$ and $[\boldsymbol{\lambda}]$ for irreducible representations of the classical groups SU$(k+1)$, SO$(2k+1)$, Sp$(2k)$ and SO$(2k)$ respectively. These have been used by Murnaghan (1938, 1958), Weyl (1939), Littlewood (1940), Wybourne (1970), and many others. The connection between these natural labels and the Dynkin labels is spelled out in table 9.

Given the simple root systems of tables 4, 5 and 8, the relation (3.4) and its inverse giving $\boldsymbol{\lambda}$ in terms of $\boldsymbol{a}$ lead to similar natural labels for the irreducible representations of

Table 9.

Algebra	Relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$		Half the sum of positive roots $\boldsymbol{\delta}$
A_k	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+a_k$	$\delta_1=k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=\quad a_2+\ldots+a_{k-1}+a_k$	$\delta_2=k-1$
	$\vdots$	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=\quad a_{k-1}+a_k$	$\delta_{k-1}=2$
	$a_k=\lambda_k$	$\lambda_k=\quad a_k$	$\delta_k=1$
B_k	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+\frac{1}{2}a_k$	$\delta_1=(2k-1)/2$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=\quad a_2+\ldots+a_{k-1}+\frac{1}{2}a_k$	$\delta_2=(2k-3)/2$
	$\vdots$	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=\quad a_{k-1}+\frac{1}{2}a_k$	$\delta_{k-1}=\frac{3}{2}$
	$a_k=2\lambda_k$	$\lambda_k=\quad \frac{1}{2}a_k$	$\delta_k=\frac{1}{2}$
C_k	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-1}+a_k$	$\delta_1=k$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=\quad a_2+\ldots+a_{k-1}+a_k$	$\delta_2=k-1$
	$\vdots$	$\vdots$	$\vdots$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=\quad a_{k-1}+a_k$	$\delta_{k-1}=2$
	$a_k=\lambda_k$	$\lambda_k=\quad a_k$	$\delta_k=1$
D_k	$a_1=\lambda_1-\lambda_2$	$\lambda_1=a_1+a_2+\ldots+a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$	$\delta_1=k-1$
	$a_2=\lambda_2-\lambda_3$	$\lambda_2=\quad a_2+\ldots+a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$	$\delta_2=k-2$
	$\vdots$	$\vdots$	$\vdots$
	$a_{k-2}=\lambda_{k-2}-\lambda_{k-1}$	$\lambda_{k-2}=\quad a_{k-2}+\frac{1}{2}a_{k-1}+\frac{1}{2}a_k$	$\delta_{k-2}=2$
	$a_{k-1}=\lambda_{k-1}-\lambda_k$	$\lambda_{k-1}=\quad \frac{1}{2}a_{k-1}+\frac{1}{2}a_k$	$\delta_{k-1}=1$
	$a_k=\lambda_{k-1}+\lambda_k$	$\lambda_k=\quad \frac{1}{2}a_{k-1}-\frac{1}{2}a_k$	$\delta_k=0$

the exceptional groups. It is at this stage that the advantages of basing the labelling on a singly supplemented Dynkin diagram become apparent. Thus, although the labels corresponding to table 4 are given in full in table 9, for the sake of simplicity, those corresponding to tables 5 and 8 are only included in table 10 if the Dynkin diagram is not doubly supplemented.

The information in tables 9, 10 and 11 regarding the relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$ has a number of important aspects. Firstly, the only constraints on the components of $\boldsymbol{\lambda}$ are those defined by the condition that each component of $\boldsymbol{a}$ is a non-negative integer. These conditions ensure that the components of $\boldsymbol{\lambda}$ are either integers or half odd integers. Secondly, the natural label for each elementary representation $\boldsymbol{\lambda}^{(i)}$ of the simple Lie algebra g may be found by setting $a_i=1$ and $a_j=0$ for $j\neq i$. These elementary representations are thus labelled by the columns of the matrix expressing $\boldsymbol{\lambda}$ in terms of $\boldsymbol{a}$.

Thirdly, it is known (Dynkin 1975b, p 356) that half the sum of the positive roots of a semi-simple Lie algebra of rank k, denoted by

$$\boldsymbol{R}=\tfrac{1}{2}\sum_{r>0}\boldsymbol{r}, \tag{4.5}$$

is the sum of the highest weights of the k elementary representations $\boldsymbol{\lambda}^{(i)}$ of g. Defining

$$\boldsymbol{\delta}=\boldsymbol{R}-\sum_{p\in\Gamma_g}R_m\boldsymbol{p}, \tag{4.6}$$

Table 10.

Exceptional algebra g	Classical subalgebra h	Relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$		Half the sum of positive roots $\boldsymbol{\delta}$
G_2	A_2	$a_1=\lambda_2$	$\lambda_2=2a_1+a_2$	$\delta_1=3$
		$a_2=\lambda_1-2\lambda_2$	$\lambda_2=a_1$	$\delta_2=1$
F_4	B_4	$a_1=\lambda_2-\lambda_3$	$\lambda_1=a_1+2a_2+\frac{3}{2}a_3+a_4$	$\delta_1=\frac{11}{2}$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+\frac{1}{2}a_3$	$\delta_2=\frac{5}{2}$
		$a_3=2\lambda_4$	$\lambda_3=a_2+\frac{1}{2}a_3$	$\delta_3=\frac{3}{2}$
		$a_4=\lambda_1-\lambda_2-\lambda_3-\lambda_4$	$\lambda_4 \quad \frac{1}{2}a_3$	$\delta_4=\frac{1}{2}$
F_4	$C_1\dot{+}C_3$	$a_1=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4)$	$\lambda_1=2a_1+3a_2+2a_3+a_4$	$\delta_1=8$
		$a_2=\lambda_4$	$\lambda_2=a_2+a_3+a_4$	$\delta_2=3$
		$a_3=\lambda_3-\lambda_4$	$\lambda_3=a_2+a_3$	$\delta_3=2$
		$a_4=\lambda_2-\lambda_3$	$\lambda_4=a_2$	$\delta_4=1$
E_6	$A_1\dot{+}A_5$	$a_1=\lambda_2-\lambda_3$	$\lambda_1=a_1+2a_2+3a_3+2a_4+a_5+2a_6$	$\delta_1=11$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+a_3+a_4+a_5$	$\delta_2=5$
		$a_3=\lambda_4-\lambda_5$	$\lambda_3=a_2+a_3+a_4+a_5$	$\delta_3=4$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_3+a_4+a_5$	$\delta_4=3$
		$a_5=\lambda_6$	$\lambda_5=a_4+a_5$	$\delta_5=2$
		$a_7=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4+\lambda_5+\lambda_6)$	$\lambda_6=a_5$	$\delta_6=1$
E_6	$A_2\dot{+}A_2\dot{+}A_2$	$a_1=\lambda_3-\lambda_4$	$\lambda_1=a_1+2a_2+3a_3+2a_4+a_5+2a_6$	$\delta_1=11$
		$a_2=\lambda_4$	$\lambda_2=a_6$	$\delta_2=1$
		$a_3=\frac{1}{3}(\lambda_1-2\lambda_2-\lambda_3-\lambda_4-2\lambda_5+\lambda_6)$	$\lambda_3=a_1+a_2$	$\delta_3=2$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_2$	$\delta_4=1$
		$a_5=\lambda_6$	$\lambda_5=a_4+a_5$	$\delta_5=2$
		$a_6=\lambda_2$	$\lambda_6=a_5$	$\delta_6=1$
E_7	A_7	$a_1=\lambda_7$	$\lambda_1=2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7$	$\delta_1=17$
		$a_2=\lambda_6-\lambda_7$	$\lambda_2=a_1+a_2+a_3+a_4+a_5+a_6$	$\delta_2=6$
		$a_3=\lambda_5-\lambda_6$	$\lambda_3=a_1+a_2+a_3+a_4+a_5$	$\delta_3=5$
		$a_4=\lambda_4-\lambda_5$	$\lambda_4=a_1+a_2+a_3+a_4$	$\delta_4=4$
		$a_5=\lambda_3-\lambda_4$	$\lambda_5=a_1+a_2+a_3$	$\delta_5=3$
		$a_6=\lambda_2-\lambda_3$	$\lambda_6=a_1+a_2$	$\delta_6=2$
		$a_7=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5+\lambda_6+\lambda_7)$	$\lambda_7=a_1$	$\delta_7=1$

E_7	$A_1 \dotplus D_6$	$a_1=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7)$	$\lambda_1=2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7$	$\delta_1=17$
		$a_2=\lambda_6+\lambda_7$	$\lambda_2=\frac{1}{2}a_2+a_3+a_4+a_5+a_6+\frac{1}{2}a_7$	$\delta_2=5$
		$a_3=\lambda_5-\lambda_6$	$\lambda_3=\frac{1}{2}a_2+a_3+a_4+a_5+\frac{1}{2}a_7$	$\delta_3=4$
		$a_4=\lambda_4-\lambda_5$	$\lambda_4=\frac{1}{2}a_2+a_3+a_4+\frac{1}{2}a_7$	$\delta_4=3$
		$a_5=\lambda_3-\lambda_4$	$\lambda_5=\frac{1}{2}a_2+a_3+\frac{1}{2}a_7$	$\delta_5=2$
		$a_6=\lambda_2-\lambda_3$	$\lambda_6=\frac{1}{2}a_2+\frac{1}{2}a_7$	$\delta_6=1$
		$a_7=\lambda_6-\lambda_7$	$\lambda_7=\frac{1}{2}a_2-\frac{1}{2}a_7$	$\delta_7=0$
E_7	$A_2 \dotplus A_5$	$a_1=\lambda_2$	$\lambda_1=2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7$	$\delta_1=17$
		$a_2=\frac{1}{3}(\lambda_1-2\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6+2\lambda_7)$	$\lambda_2=a_1$	$\delta_2=1$
		$a_3=\lambda_6-\lambda_7$	$\lambda_3=a_3+a_4+a_5+a_6+a_7$	$\delta_3=5$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_3+a_4+a_5+a_7$	$\delta_4=4$
		$a_5=\lambda_4-\lambda_5$	$\lambda_5=a_3+a_4+a_7$	$\delta_5=3$
		$a_6=\lambda_3-\lambda_4$	$\lambda_6=a_3+a_7$	$\delta_6=2$
		$a_7=\lambda_7$	$\lambda_7=a_7$	$\delta_7=1$
E_8	A_8	$a_1=\lambda_8$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\lambda_7-\lambda_8$	$\lambda_2=a_1+a_2+a_3+a_4+a_5+a_6+a_7$	$\delta_2=7$
		$a_3=\lambda_6-\lambda_7$	$\lambda_3=a_1+a_2+a_3+a_4+a_5+a_6$	$\delta_3=6$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_1+a_2+a_3+a_4+a_5$	$\delta_4=5$
		$a_5=\lambda_4-\lambda_5$	$\lambda_5=a_1+a_2+a_3+a_4$	$\delta_5=4$
		$a_6=\lambda_3-\lambda_4$	$\lambda_6=a_1+a_2+a_3$	$\delta_6=3$
		$a_7=\lambda_2-\lambda_3$	$\lambda_7=a_1+a_2$	$\delta_7=2$
		$a_8=\frac{1}{3}(\lambda_1-2\lambda_2-2\lambda_3-2\lambda_4+\lambda_5+\lambda_6+\lambda_7+\lambda_8)$	$\lambda_8=a_1$	$\delta_8=1$
E_8	D_8	$a_1=\lambda_2-\lambda_3$	$\lambda_1=a_1+2a_2+3a_3+4a_4+5a_5+\frac{7}{2}a_6+2a_7+\frac{5}{2}a_8$	$\delta_1=23$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+a_3+a_4+a_5+\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_2=6$
		$a_3=\lambda_4-\lambda_5$	$\lambda_3=a_2+a_3+a_4+a_5+\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_3=5$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_3+a_4+a_5+\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_4=4$
		$a_5=\lambda_6-\lambda_7$	$\lambda_5=a_4+a_5+\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_5=3$
		$a_6=\lambda_7-\lambda_8$	$\lambda_6=a_5+\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_6=2$
		$a_7=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5-\lambda_6-\lambda_7+\lambda_8)$	$\lambda_7=\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_7=1$
		$a_8=\lambda_7+\lambda_8$	$\lambda_8=-\frac{1}{2}a_6+\frac{1}{2}a_8$	$\delta_8=0$

Table 10—continued

Exceptional algebra g	Classical subalgebra h	Relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$		Half the sum of positive roots $\boldsymbol{\delta}$
E_8	$A_7 \dotplus A_1$	$a_1=\lambda_7$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\lambda_6-\lambda_7$	$\lambda_2=a_1+a_2+a_3+a_4+a_5+a_8$	$\delta_2=6$
		$a_3=\lambda_5-\lambda_6$	$\lambda_3=a_1+a_2+a_3+a_4+a_5$	$\delta_3=5$
		$a_4=\lambda_4-\lambda_5$	$\lambda_4=a_1+a_2+a_3+a_4$	$\delta_4=4$
		$a_5=\lambda_3-\lambda_4$	$\lambda_5=a_1+a_2+a_3$	$\delta_5=3$
		$a_6=\frac{1}{4}(\lambda_1-3\lambda_2-3\lambda_3+\lambda_4+\lambda_5+\lambda_6+\lambda_7-2\lambda_8)$	$\lambda_6=a_1+a_2$	$\delta_6=2$
		$a_7=\lambda_8$	$\lambda_7=a_1$	$\delta_7=1$
		$a_8=\lambda_2-\lambda_3$	$\lambda_8=a_7$	$\delta_8=1$
E_8	$A_4 \dotplus A_4$	$a_1=\lambda_4$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+a_3$	$\delta_2=3$
		$a_3=\lambda_2-\lambda_3$	$\lambda_3=a_1+a_2$	$\delta_3=2$
		$a_4=\frac{1}{5}(\lambda_1-4\lambda_2+\lambda_3+\lambda_4-2\lambda_5-2\lambda_6-2\lambda_7+3\lambda_8)$	$\lambda_4=a_1$	$\delta_4=1$
		$a_5=\lambda_7-\lambda_8$	$\lambda_5=a_5+a_6+a_7+a_8$	$\delta_5=4$
		$a_6=\lambda_6-\lambda_7$	$\lambda_6=a_5+a_6+a_8$	$\delta_6=3$
		$a_7=\lambda_5-\lambda_6$	$\lambda_7=a_5+a_8$	$\delta_7=2$
		$a_8=\lambda_8$	$\lambda_8=a_8$	$\delta_8=1$
E_8	$A_5 \dotplus A_2 \dotplus A_1$	$a_1=\lambda_5$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\lambda_4-\lambda_5$	$\lambda_2=a_1+a_2+a_3+a_4$	$\delta_2=4$
		$a_3=\lambda_3-\lambda_4$	$\lambda_3=a_1+a_2+a_3$	$\delta_3=3$
		$a_4=\lambda_2-\lambda_3$	$\lambda_4=a_1+a_2$	$\delta_4=2$
		$a_5=\frac{1}{6}(\lambda_1-5\lambda_2+\lambda_3+\lambda_4+\lambda_5-4\lambda_6+2\lambda_7-3\lambda_8)$	$\lambda_5=a_1$	$\delta_5=1$
		$a_6=\lambda_6-\lambda_7$	$\lambda_6=a_6+a_7$	$\delta_6=2$
		$a_7=\lambda_7$	$\lambda_7=a_7$	$\delta_7=1$
		$a_8=\lambda_8$	$\lambda_8=a_8$	$\delta_8=1$
E_8	$A_2 \dotplus A_2 \dotplus A_2 \dotplus A_2$	$a_1=\lambda_2$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\frac{1}{3}(\lambda_1-2\lambda_2-2\lambda_3+\lambda_4-\lambda_7-\lambda_8)$	$\lambda_2=a_1$	$\delta_2=1$
		$a_3=\lambda_8$	$\lambda_3=a_3+2a_4+3a_5+2a_6+a_7+2a_8$	$\delta_3=11$
		$a_4=\lambda_7-\lambda_8$	$\lambda_4=a_8$	$\delta_4=1$
		$a_5=\frac{1}{3}(\lambda_3-2\lambda_4-\lambda_5-\lambda_5-\lambda_6-2\lambda_7+\lambda_8)$	$\lambda_5=a_6+a_7$	$\delta_5=2$
		$a_6=\lambda_6$	$\lambda_6=a_6$	$\delta_6=1$
		$a_7=\lambda_5-\lambda_6$	$\lambda_7=a_3+a_4$	$\delta_7=2$
		$a_8=\lambda_4$	$\lambda_8=a_3$	$\delta_8=1$

Table 11.

Exceptional algebra g	Classical subalgebra h	Relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$		Half the sum of positive roots $\boldsymbol{\lambda}$
E_6	$A_5 \dot{+} A_1$	$a_1=\lambda_2-\lambda_3$	$\lambda_1=a_1+2a_2+3a_3+2a_4+a_5+2a_6$	$\delta_1=11$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+a_3+a_6$	$\delta_2=4$
		$a_3=\lambda_4-\lambda_5$	$\lambda_3=a_2+a_3+a_6$	$\delta_3=3$
		$a_4=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4+\lambda_5-\lambda_6)$	$\lambda_4=a_3+a_6$	$\delta_4=2$
		$a_5=\lambda_6$	$\lambda_5=a_6$	$\delta_5=1$
		$a_6=\lambda_5$	$\lambda_6=a_5$	$\delta_6=1$
E_7	$D_6 \dot{+} A_1$	$a_1=\lambda_2-\lambda_3$	$\lambda_1=a_1+2a_2+3a_3+\frac{5}{2}a_4+2a_5+a_6+\frac{3}{2}a_7$	$\delta_1=13$
		$a_2=\lambda_3-\lambda_4$	$\lambda_2=a_1+a_2+a_3+\frac{1}{2}a_4+\frac{1}{2}a_7$	$\delta_2=4$
		$a_3=\lambda_4-\lambda_5$	$\lambda_3=a_2+a_3+\frac{1}{2}a_4+\frac{1}{2}a_7$	$\delta_3=3$
		$a_4=\lambda_5-\lambda_6$	$\lambda_4=a_3+\frac{1}{2}a_4+\frac{1}{2}a_7$	$\delta_4=2$
		$a_5=\frac{1}{2}(\lambda_1-\lambda_2-\lambda_3-\lambda_4-\lambda_5+\lambda_6-\lambda_7)$	$\lambda_5=\frac{1}{2}a_4+\frac{1}{2}a_7$	$\delta_5=1$
		$a_6=\lambda_7$	$\lambda_6=-\frac{1}{2}a_4+\frac{1}{2}a_7$	$\delta_6=0$
		$a_7=\lambda_5+\lambda_6$	$\lambda_7=a_6$	$\delta_7=1$
E_7	$A_5 \dot{+} A_2$	$a_1=\lambda_5$	$\lambda_1=2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7$	$\delta_1=17$
		$a_2=\lambda_4-\lambda_5$	$\lambda_2=a_1+a_2+a_3+a_7$	$\delta_2=4$
		$a_3=\lambda_3-\lambda_4$	$\lambda_3=a_1+a_2+a_3$	$\delta_3=3$
		$a_4=\frac{1}{3}(\lambda_1-2\lambda_2-2\lambda_3+\lambda_4+\lambda_5-\lambda_6-\lambda_7)$	$\lambda_4=a_1+a_2$	$\delta_4=2$
		$a_5=\lambda_7$	$\lambda_5=a_1$	$\delta_5=1$
		$a_6=\lambda_6-\lambda_7$	$\lambda_6=a_5+a_6$	$\delta_6=2$
		$a_7=\lambda_2-\lambda_3$	$\lambda_7=a_5$	$\delta_7=1$

Table 11—continued

Exceptional algebra g	Classical subalgebra h	Relationship between the Dynkin label $\boldsymbol{a}$ and the natural label $\boldsymbol{\lambda}$		Half the sum of positive roots $\boldsymbol{\lambda}$
E_7	$A_3\dot{+}A_3\dot{+}A_1$	$a_1=\lambda_3$	$\lambda_1=2a_1+3a_2+4a_3+3a_4+2a_5+a_6+2a_7$	$\delta_1=17$
		$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_1+a_2$	$\delta_2=2$
		$a_3=\frac{1}{4}(\lambda_1-3\lambda_2+\lambda_3-\lambda_4-\lambda_5-\lambda_6-2\lambda_7)$	$\lambda_3=a_1$	$\delta_3=1$
		$a_4=\lambda_6$	$\lambda_4=a_4+a_5+a_6$	$\delta_4=3$
		$a_5=\lambda_5-\lambda_6$	$\lambda_5=a_4+a_5$	$\delta_5=2$
		$a_6=\lambda_4-\lambda_5$	$\lambda_6=a_4$	$\delta_6=1$
		$a_7=\lambda_7$	$\lambda_7=a_7$	$\delta_7=1$
E_8	$A_3\dot{+}D_5$	$a_1=\lambda_3$	$\lambda_1=2a_1+3a_2+4a_3+5a_4+6a_5+4a_6+2a_7+3a_8$	$\delta_1=29$
		$a_2=\lambda_2-\lambda_3$	$\lambda_2=a_1+a_2$	$\delta_2=2$
		$a_3=\frac{1}{4}(\lambda_1-3\lambda_2+\lambda_3-2\lambda_4-2\lambda_5-2\lambda_6-2\lambda_7+2\lambda_8)$	$\lambda_3=a_1$	$\delta_3=1$
		$a_4=\lambda_7-\lambda_8$	$\lambda_4=\frac{1}{2}a_4+a_5+a_6+a_7+\frac{1}{2}a_8$	$\delta_4=4$
		$a_5=\lambda_6-\lambda_7$	$\lambda_5=\frac{1}{2}a_4+a_5+a_6+\frac{1}{2}a_8$	$\delta_5=3$
		$a_6=\lambda_5-\lambda_6$	$\lambda_6=\frac{1}{2}a_4+a_5+\frac{1}{2}a_8$	$\delta_6=2$
		$a_7=\lambda_4-\lambda_5$	$\lambda_7=\frac{1}{2}a_4+\frac{1}{2}a_8$	$\delta_7=1$
		$a_8=\lambda_7+\lambda_8$	$\lambda_8=-\frac{1}{2}a_4+\frac{1}{2}a_8$	$\delta_8=0$

this implies that

$$\boldsymbol{\delta} = \sum_{i=1}^{k} \boldsymbol{\lambda}^{(i)}, \tag{4.7}$$

yielding the results given in the final columns of tables 9, 10 and 11.

Finally, the tables may be used to write down the modification rules and equivalence relations appropriate to the characters of irreducible representations of semi-simple Lie groups. They are a consequence of the character formula due to Weyl (1926), which may be written in the form

$$\chi^{\lambda}(\boldsymbol{\phi}) = \frac{\sum_{S \in \mathrm{W}_G} \eta_S \, \mathrm{e}^{\mathrm{i}S(\boldsymbol{R}+\boldsymbol{M}) \cdot \boldsymbol{\phi}}}{\sum_{S \in \mathrm{W}_G} \eta_S \, \mathrm{e}^{\mathrm{i}S\boldsymbol{R} \cdot \boldsymbol{\phi}}} \tag{4.8}$$

where $\boldsymbol{\phi} = (\phi_1, \phi_2, \ldots, \phi_d)$ is a set of real class parameters, $\boldsymbol{M}$ is the highest weight vector of the irreducible representation $\boldsymbol{\lambda}$ of the Lie group G, $\boldsymbol{R}$ is half the sum of the positive roots of the corresponding Lie algebra g, S is an element of the Weyl symmetry group W_G of G and η_S is the parity of S, that is $\eta_S = +1$ or -1 according to whether the number of Weyl reflections generating S is even or odd. Associated with each root $\boldsymbol{r}$ of the Lie algebra g there exists a Weyl reflection S_r in the hyperplane perpendicular to $\boldsymbol{r}$. This reflection takes place in the Euclidean space W, and is defined by

$$S_r : \boldsymbol{v} \to \boldsymbol{v} - \frac{2(\boldsymbol{v} \cdot \boldsymbol{r})\boldsymbol{r}}{(\boldsymbol{r} \cdot \boldsymbol{r})} \tag{4.9}$$

for any vector $\boldsymbol{v}$ in W. It should be noticed that by virtue of (2.1) each vector $\boldsymbol{p}$ of $V^{\perp}$ is left invariant by such reflections.

It follows from (4.8) that if

$$\boldsymbol{\lambda} = \boldsymbol{M} - \sum_{\boldsymbol{p} \in \Gamma_g} M_m \boldsymbol{p} \tag{4.10}$$

and

$$\boldsymbol{\mu} = \boldsymbol{N} - \sum_{\boldsymbol{p} \in \Gamma_g} N_m \boldsymbol{p} \tag{4.11}$$

with

$$\boldsymbol{M} = S(\boldsymbol{R} + \boldsymbol{N}) - \boldsymbol{R}, \tag{4.12}$$

then

$$\chi^{\lambda}(\boldsymbol{\phi}) = \eta_S \chi^{\mu}(\boldsymbol{\phi}).$$

This relationship between the characters of $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ defines an equivalence relation between irreducible representations. The rule by which $\boldsymbol{\lambda}$ is obtained from $\boldsymbol{\mu}$ using (4.9)–(4.12) and (2.1) is known as a modification rule. Its use is to provide a means whereby a non-standard, inadmissible label $\boldsymbol{\mu}$ may be replaced by a standard label $\boldsymbol{\lambda}$ satisfying the constraints imposed in tables 9, 10 and 11 by the requirement that the components of $\boldsymbol{a}$ are all non-negative integers. Such non-standard labels $\boldsymbol{\mu}$ arise, for example, in the evaluation of Kronecker products of irreducible representations in accordance with the method of Racah (1964) and Speiser (1964). In such a context there always exists some S in W_G such that for each non-standard label $\boldsymbol{\mu}$ a standard label $\boldsymbol{\lambda}$ exists, unless

$$\chi^{\mu}(\boldsymbol{\phi}) = 0.$$

This condition only arises if there exists a reflection S_r such that

$$\chi^{\mu}(\boldsymbol{\phi})=\eta_{S_r}\chi^{\mu}(\boldsymbol{\phi})=-\chi^{\mu}(\boldsymbol{\phi}).$$

The complete set of Weyl reflections, and thus the whole Weyl group, may be generated by means of the reflections in the hyperplanes perpendicular to the simple roots. It is thus straightforward, using the tables presented earlier, to derive the modification rules defined by

$$\chi^{\lambda}(\boldsymbol{\phi})=\eta_{S_r}\chi^{\mu}(\boldsymbol{\phi})=-\chi^{\mu}(\boldsymbol{\phi})$$

for each simple root $\boldsymbol{r}$. These rules are displayed in tables 12 and 13 for each of the labelling schemes of tables 9 and 10. Similar results may easily be established for the schemes of table 11.

5. Conclusion

The tables presented here provide a systematic account of the natural labelling schemes for both the simple root systems and the irreducible representations of the exceptional Lie algebras. These schemes are based on the relationship between these algebras and classical semi-simple subalgebras of the same rank. In almost all cases a labelling may be chosen which differs from a classical labelling in just one particular. This is accomplished through the use of the schemes based on singly supplemented Dynkin diagrams. Furthermore, an ordering of basis vectors in the Euclidean space W in which the root and weight spaces are embedded has been chosen so that in the natural

Table 12.

Algebra g	$\boldsymbol{\mu}$	$\boldsymbol{\lambda}$ where $\chi^{\lambda}(\boldsymbol{\phi})=\eta_{S_r}\chi^{\mu}(\boldsymbol{\phi})=-\chi^{\mu}(\boldsymbol{\phi})$ with $\boldsymbol{r}\in\Pi_g$				
A_k	μ_1	μ_2-1	μ_1	$\dots$	μ_1	$\mu_1-\mu_k-1$
	μ_2	μ_1+1	μ_3-1		μ_2	$\mu_2-\mu_k-1$
	μ_3	μ_3	μ_2+1		μ_3	$\mu_3-\mu_k-1$
	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$
	μ_{k-1}	μ_{k-1}	μ_{k-1}		μ_k-1	$\mu_{k-1}-\mu_k-1$
	μ_k	μ_k	μ_k		$\mu_{k-1}+1$	$-\mu_k-2$
B_4	μ_1					μ_1
	μ_2					μ_2
	$\vdots$					$\vdots$
	μ_{k-1}	,,	,,		,,	μ_{k-1}
	μ_k					$-\mu_k-1$
C_k	μ_1					μ_1
	μ_2					μ_2
	$\vdots$	,,	,,		,,	$\vdots$
	μ_{k-1}					μ_{k-1}
	μ_k					$-\mu_k-2$
D_k	μ_1					μ_1
	μ_2					μ_2
	$\vdots$	,,	,,		,,	$\vdots$
	μ_{k-1}					$-\mu_k-1$
	μ_k					$-\mu_{k-1}-1$

Table 13.

Exceptional algebra g	Classical subalgebra h	$\boldsymbol{\mu}$	$\boldsymbol{\lambda}$ where $\chi^{\boldsymbol{\lambda}}(\boldsymbol{\phi}) = \eta_{S_r}\chi^{\boldsymbol{\mu}}(\boldsymbol{\phi}) = -\chi^{\boldsymbol{\mu}}(\boldsymbol{\phi})$ with $\boldsymbol{r} \in \Pi_g$						
G_2	A_2	μ_1	$\mu_1-\mu_2-1$	μ_1					
		μ_2	$-\mu_2-2$	$\mu_2+\epsilon$					$\epsilon=\mu_1-2\mu_2+1$
F_4	B_4	μ_1	μ_1	μ_1	μ_1	$\mu_1-\epsilon$			
		μ_2	μ_3-1	μ_2	μ_2	$\mu_2+\epsilon$			
		μ_3	μ_2+1	μ_4-1	μ_3	$\mu_3+\epsilon$			$\epsilon=\frac{1}{2}(\mu_1-\mu_2-\mu_3-\mu_4+1)$
		μ_4	μ_4	μ_3+1	$-\mu_4-1$	$\mu_4+\epsilon$			
F_4	$C_1\dot{+}C_3$	μ_1	μ_1	μ_1	μ_1	$\mu_1-\epsilon$			
		μ_2	μ_3-1	μ_2	μ_2	$\mu_2+\epsilon$			
		μ_3	μ_2+1	μ_4-1	μ_3	$\mu_3+\epsilon$			$\epsilon=\frac{1}{2}(\mu_1-\mu_2-\mu_3-\mu_4+2)$
		μ_4	μ_4	μ_3+1	$-\mu_4-2$	$\mu_4+\epsilon$			
E_6	$A_1\dot{+}A_5$	μ_1	μ_1	μ_1	μ_1	μ_1	μ_1	$\mu_1-\epsilon$	
		μ_2	μ_3-1	μ_2	μ_2	μ_2	$\mu_2-\mu_6-1$	$\mu_2+\epsilon$	
		μ_3	μ_2+1	μ_4-1	μ_3	μ_3	$\mu_3-\mu_6-1$	$\mu_3+\epsilon$	$\epsilon=\frac{1}{2}(\mu_1-\mu_2-\mu_3-\mu_4+\mu_5+\mu_6+2)$
		μ_4	μ_4	μ_3+1	μ_5-1	μ_4	$\mu_4-\mu_6-1$	$\mu_4+\epsilon$	
		μ_5	μ_5	μ_5	μ_4+1	μ_6-1	$\mu_5-\mu_6-1$	μ_5	
		μ_6	μ_6	μ_6	μ_6	μ_5+1	$-\mu_6-2$	μ_6	
E_6	$A_2\dot{+}A_2\dot{+}A_2$	μ_1	$\mu_1-\mu_2-1$	μ_1	μ_1	μ_1	μ_1	μ_1	
		μ_2	$-\mu_2-2$	μ_2	μ_2	μ_2	μ_2	$\mu_2+\epsilon$	
		μ_3	μ_4	μ_4-1	$\mu_3-\mu_4-1$	μ_3	μ_3	$\mu_3+\epsilon$	$\epsilon=\frac{1}{3}(\mu_1-2\mu_2-\mu_3-\mu_4$
		μ_4	μ_3	μ_3+1	$-\mu_4-2$	μ_4	μ_4	$\mu_4+\epsilon$	$-2\mu_5+\mu_6+3)$
		μ_5	μ_5	μ_5	μ_5	μ_6-1	$\mu_5-\mu_6-1$	$\mu_5+\epsilon$	
		μ_6	μ_6	μ_6	μ_6	μ_5+1	$-\mu_6-2$	μ_6	
E_7	A_7	μ_1	μ_1	μ_1	$\ldots$	μ_1	$\mu_1-\mu_7-1$	μ_1	
		μ_2	μ_3-1	μ_2		μ_2	$\mu_2-\mu_7-1$	$\mu_2+\epsilon$	
		μ_3	μ_2+1	μ_4-1		μ_3	$\mu_3-\mu_7-1$	$\mu_3+\epsilon$	
		μ_4	μ_4	μ_3+1		μ_4	$\mu_4-\mu_7-1$	$\mu_4+\epsilon$	$\epsilon=\frac{1}{2}(\mu_1-\mu_2-\mu_3-\mu_4-\mu_5$
		μ_5	μ_5	μ_5		μ_5	$\mu_5-\mu_7-1$	$\mu_5+\epsilon$	$+\mu_6+\mu_7+2)$
		μ_6	μ_6	μ_6		μ_7-1	$\mu_6-\mu_7-1$	μ_6	
		μ_7	μ_7	μ_7		μ_6+1	$-\mu_7-2$	μ_7	

Table 13—continued

Exceptional algebra g	Classical subalgebra h	$\boldsymbol{\mu}$	$\boldsymbol{\lambda}$ where $\chi^{\boldsymbol{\lambda}}(\boldsymbol{\phi})=\eta_{S_r}\chi^{\boldsymbol{\mu}}(\boldsymbol{\phi})=-\chi^{\boldsymbol{\mu}}(\boldsymbol{\phi})$ with $\boldsymbol{r}\in\Pi_g$							
E_7	$A_1\dot{+}D_6$	μ_1	μ_1	$\dots$	μ_1	μ_1	μ_1			
		μ_2	μ_3-1		μ_2	μ_2	$\mu_2-2\epsilon$			
		μ_3	μ_2+1		μ_3	μ_3	$\mu_3+\epsilon$			$\epsilon=\frac{1}{2}(\mu_1-\mu_2-\mu_3-\mu_4-\mu_5-\mu_6+\mu_7+2)$
		μ_4	μ_4		μ_4	μ_4	$\mu_4+\epsilon$			
		μ_6	μ_6		μ_7-1	$-\mu_7-1$	$\mu_6+\epsilon$			
		μ_7	μ_7		μ_6+1	$-\mu_6-1$	$\mu_7-\epsilon$			
E_7	$A_2\dot{+}A_5$	μ_1	$\mu_1-\mu_2-1$	μ_1	μ_1	$\dots$	μ_1	μ_1	μ_1	
		μ_2	$-\mu_2-2$	μ_2	μ_2		μ_2	μ_2	$\mu_2+\epsilon$	
		μ_3	μ_3	μ_4-1	μ_3		μ_3	$\mu_3-\mu_7-1$	$\mu_3+\epsilon$	
		μ_4	μ_4	μ_3+1	μ_5-1		μ_4	$\mu_4-\mu_7-1$	$\mu_4+\epsilon$	$\epsilon=\frac{1}{3}(\mu_1-2\mu_2-\mu_3-\mu_4-\mu_5-\mu_6+2\mu_7+3)$
		μ_5	μ_5	μ_5	μ_4+1		μ_5	$\mu_5-\mu_7-1$	$\mu_5+\epsilon$	
		μ_6	μ_6	μ_6	μ_6		μ_7-1	$\mu_6-\mu_7-1$	$\mu_6+\epsilon$	
		μ_7	μ_7	μ_7	μ_7		μ_6+1	$-\mu_7-2$	μ_7	
E_8	A_8	μ_1	μ_1	$\dots$	μ_1	$\mu_1-\mu_8-1$	μ_1			
		μ_2	μ_3-1		μ_2	$\mu_2-\mu_8-1$	$\mu_2+\epsilon$			
		μ_3	μ_2+1		μ_3	$\mu_3-\mu_8-1$	$\mu_3+\epsilon$			
		μ_4	μ_4		μ_4	$\mu_4-\mu_8-1$	$\mu_4+\epsilon$			$\epsilon=\frac{1}{3}(\mu_1-2\mu_2-2\mu_3-2\mu_4+\mu_5+\mu_6+\mu_7+\mu_8+3)$
		μ_5	μ_5		μ_5	$\mu_5-\mu_8-1$	μ_5			
		μ_6	μ_6		μ_6	$\mu_6-\mu_8-1$	μ_6			
		μ_7	μ_7		μ_8-1	$\mu_7-\mu_8-1$	μ_7			
		μ_8	μ_8		μ_7+1	$-\mu_8-2$	μ_8			
E_8	D_8	μ_1	μ_1	$\dots$	μ_1	μ_1	$\mu_1-\epsilon$			
		μ_2	μ_3-1		μ_2	μ_2	$\mu_2+\epsilon$			
		μ_3	μ_2+1		μ_3	μ_3	$\mu_3+\epsilon$			
		μ_4	μ_4		μ_4	μ_4	$\mu_4+\epsilon$			$\epsilon=\frac{1}{4}(\mu_1-\mu_2-\mu_3-\mu_4-\mu_5-\mu_6-\mu_7+\mu_8+2)$
		μ_5	μ_5		μ_5	μ_5	$\mu_5+\epsilon$			
		μ_6	μ_6		μ_6	μ_6	$\mu_6+\epsilon$			
		μ_7	μ_7		μ_8-1	$-\mu_8-1$	$\mu_7+\epsilon$			
		μ_8	μ_8		μ_7+1	$-\mu_7-1$	$\mu_8-\epsilon$			

E_8	$A_7 \dotplus A_1$	μ_1	μ_1	$\ldots$	μ_1	$\mu_1-\mu_7-1$	μ_1				
		μ_2	μ_3-1		μ_2	$\mu_2-\mu_7-1$	$\mu_2-\epsilon$				
		μ_3	μ_2+1		μ_3	$\mu_3-\mu_7-1$	$\mu_3-\epsilon$				
		μ_4	μ_4		μ_4	$\mu_4-\mu_7-1$	μ_4				$\epsilon=\frac{1}{4}(\mu_1-3\mu_2-3\mu_3+\mu_4+\mu_5+\mu_6$
		μ_5	μ_5		μ_5	$\mu_5-\mu_7-1$	μ_5				$+\mu_7-2\mu_8+4)$
		μ_6	μ_6		μ_7-1	$\mu_6-\mu_7-1$	μ_6				
		μ_7	μ_7		μ_6+1	$-\mu_7-2$	μ_7				
		μ_8	μ_8		μ_8	μ_8	$\mu_8-\epsilon$				
E_8	$A_4 \dotplus A_4$	μ_1	μ_1	μ_1	$\mu_1-\mu_4-1$	μ_1	μ_1	μ_1	μ_1	μ_1	
		μ_2	μ_3-1	μ_2	$\mu_2-\mu_4-1$	μ_2	μ_2	μ_2	μ_2	$\mu_2-\epsilon$	
		μ_3	μ_2+1	μ_4-1	$\mu_3-\mu_4-1$	μ_3	μ_3	μ_3	μ_3	μ_3	
		μ_4	μ_4	μ_3+1	$-\mu_4-2$	μ_4	μ_4	μ_4	μ_4	μ_4	$\epsilon=\frac{1}{5}(\mu_1-4\mu_2+\mu_3+\mu_4-2\mu_5-2\mu_6$
		μ_5	μ_5	μ_5	μ_5	μ_6-1	μ_5	μ_5	$\mu_5-\mu_8-1$	$\mu_5-\epsilon$	$-2\mu_7+3\mu_8+5)$
		μ_6	μ_6	μ_6	μ_6	μ_5+1	μ_7-1	μ_6	$\mu_6-\mu_8-1$	$\mu_6-\epsilon$	
		μ_7	μ_7	μ_7	μ_7	μ_7	μ_6+1	μ_8-1	$\mu_7-\mu_8-1$	$\mu_7-\epsilon$	
		μ_8	μ_8	μ_8	μ_8	μ_8	μ_8	μ_7+1	$-\mu_8-2$	μ_8	
E_8	$A_5 \dotplus A_2 \dotplus A_1$	μ_1	μ_1	$\ldots$	μ_1	$\mu_1-\mu_5-1$	μ_1	μ_1	μ_1	μ_1	
		μ_2	μ_3-1		μ_2	$\mu_2-\mu_5-1$	μ_2	μ_2	μ_2	$\mu_2-\epsilon$	
		μ_3	μ_2+1		μ_3	$\mu_3-\mu_5-1$	μ_3	μ_3	μ_3	μ_3	
		μ_4	μ_4		μ_5-1	$\mu_4-\mu_5-1$	μ_4	μ_4	μ_4	μ_4	$\epsilon=\frac{1}{6}(\mu_1-5\mu_2+\mu_3+\mu_4+\mu_5-4\mu_6$
		μ_5	μ_5		μ_4+1	$-\mu_5-2$	μ_5	μ_5	μ_5	μ_5	$+2\mu_7-3\mu_8+6)$
		μ_6	μ_6		μ_6	μ_6	μ_7-1	$\mu_6-\mu_7-1$	μ_6	$\mu_6-\epsilon$	
		μ_7	μ_7		μ_7	μ_7	μ_6+1	$-\mu_7-2$	μ_7	μ_7	
		μ_8	μ_8		μ_8	μ_8	μ_8	μ_8	$-\mu_8-2$	$\mu_8-\epsilon$	
E_8	$A_2 \dotplus A_2 \dotplus$	μ_1	$\mu_1-\mu_2-1$	μ_1	μ_1	μ_1	μ_1	μ_1	μ_1	μ_1	
	$A_2 \dotplus A_2$	μ_2	$-\mu_2-2$	μ_2	μ_2	μ_2	μ_2	μ_2	$\mu_2-\epsilon_1$	μ_2	
		μ_3	μ_3	$\mu_3-\mu_4-1$	μ_3	μ_3	μ_3	μ_3	$\mu_3-\epsilon_1$	μ_3	$\epsilon_1=\frac{1}{3}(\mu_1-2\mu_2-2\mu_3+\mu_4$
		μ_4	μ_4	$-\mu_4-2$	μ_4	μ_4	μ_4	μ_4	μ_4	$\mu_4-\epsilon_2$	$-\mu_7-\mu_8+3)$
		μ_5	μ_5	μ_5	μ_6-1	$\mu_5-\mu_6-1$	μ_5	μ_5	μ_5	$\mu_5-\epsilon_2$	$\epsilon_2=\frac{1}{3}(\mu_3-2\mu_4-\mu_5-\mu_6$
		μ_6	μ_6	μ_6	μ_5+1	$-\mu_6-2$	μ_6	μ_6	μ_6	$\mu_6-\epsilon_2$	$-2\mu_7+\mu_8+3)$
		μ_7	μ_7	μ_7	μ_7	μ_7	μ_8-1	$\mu_7-\mu_8-1$	$\mu_7-\epsilon_1$	$\mu_7-\epsilon_2$	
		μ_8	μ_8	μ_8	μ_8	μ_8	μ_7+1	$-\mu_8-2$	$\mu_8-\epsilon_1$	μ_8	

Table 14.

Simple root systems of isomorphic Lie algebras

$A_1 \approx B_1 \approx C_1$

$\circ\ e_1 - e_2$	$\bullet\ e'_1$	$\circ\ 2e''_1$

$e_1 = \frac{1}{2}(e'_1 + e'_2) = e''_1 + e''_2$ $\qquad$ $e'_1 = e_1 - e_2 = 2e''_1$

$e_2 = \frac{1}{2}(-e'_1 + e'_2) = -e''_1 + e''_2$ $\qquad$ $e'_2 = e_1 + e_2 = 2e''_2$

$B_2 \approx C_2$

 $e_1 - e_2$, e_2 $\qquad$  $e'_1 - e'_2$, $2e'_2$

$e_1 = e'_1 + e'_2$ $\qquad$ $e'_1 = \frac{1}{2}(e_1 + e_2)$

$e_2 = e'_1 - e'_2$ $\qquad$ $e'_2 = \frac{1}{2}(e_1 - e_2)$

$A_1 \dot{+} A_1 \approx D_2$

$\circ\ e_1 - e_2$	$\circ\ e'_1 - e'_2$
$\circ\ e_3 - e_4$	$\circ\ e'_1 + e'_2$

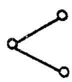
$e_1 = \frac{1}{2}(e'_1 - e'_2 + e'_3 - e'_4)$ $\qquad$ $e'_1 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4)$

$e_2 = \frac{1}{2}(-e'_1 + e'_2 + e'_3 - e'_4)$ $\qquad$ $e'_2 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4)$

$e_3 = \frac{1}{2}(e'_1 + e'_2 + e'_3 + e'_4)$ $\qquad$ $e'_3 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$

$e_4 = \frac{1}{2}(-e'_1 - e'_2 + e'_3 + e'_4)$ $\qquad$ $e'_4 = \frac{1}{2}(-e_1 - e_2 + e_3 + e_4)$

$A_3 \approx D_3$

 $e_1 - e_2$, $e_2 - e_3$, $e_3 - e_4$ $\qquad$ $e'_1 - e'_2$  $e'_2 - e'_3$, $e'_2 + e'_3$

$e_1 = \frac{1}{2}(e'_1 + e'_2 - e'_3 + e'_4)$ $\qquad$ $e'_1 = \frac{1}{2}(e_1 + e_2 - e_3 - e_4)$

$e_2 = \frac{1}{2}(e'_1 - e'_2 + e'_3 + e'_4)$ $\qquad$ $e'_2 = \frac{1}{2}(e_1 - e_2 + e_3 - e_4)$

$e_3 = \frac{1}{2}(-e'_1 + e'_2 + e'_3 + e'_4)$ $\qquad$ $e'_3 = \frac{1}{2}(-e_1 + e_2 + e_3 - e_4)$

$e_4 = \frac{1}{2}(-e'_1 - e'_2 - e'_3 + e'_4)$ $\qquad$ $e'_4 = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$

labelling schemes the last $(d-k)$ components of the irreducible representation label $\boldsymbol{\lambda}$ vanish, whilst for the exceptional algebra g only the first component λ_1 of $\boldsymbol{\lambda}$ differs in its range of allowed value from the corresponding labels of the irreducible representations of the classical subalgebra h. Moreover, the requirement for a sound labelling scheme, emphasised by Wybourne and Bowick (1977), that on restriction from g to h

$$\boldsymbol{\lambda} \rightarrow \boldsymbol{\lambda} + \ldots, \tag{5.1}$$

is automatically satisfied.

Further schemes may be devised if changes of scale are tolerated which would violate (5.1) unless scale changes are incorporated into a new definition of $\boldsymbol{\lambda}$ which would replace (4.3). For example, in the case of F_4 a natural scheme has been based here on the subalgebra $C_1 \dotplus C_3$. No such scheme may be based on the isomorphic algebras $A_1 \dotplus C_3$ or $B_1 \dotplus C_3$. This is because the root spaces of A_1, B_1 and C_1 are distinguished by changes of scale. This is made clear in table 4, along with similar distinctions involving the isomorphic pairs of algebras $B_2 \approx C_2$, $D_2 \approx A_1 \dotplus A_1$ and $A_3 \approx D_3$. The ratio of lengths of simple roots in A_1, B_1 and C_1 is $\sqrt{2}:1:2$ and in B_2 and C_2 is $\sqrt{2}:2$. However, in the case of the pair D_2 and $A_1 \dotplus A_1$ and the pair A_3 and D_3 the ratios are $1:1$, that is, there are no changes of scale involved. Thus in tables 7 and 8 natural labelling schemes may be devised based on D_3 rather than A_3, and upon D_2 rather than $A_1 \dotplus A_1$. These are not given here, even though they do serve to illustrate very clearly the chains (3.4)–(3.7) by virtue of their dependence upon the well known embeddings

$$D_6 \supset D_3 \dotplus D_3, \tag{5.2}$$

$$D_8 \supset D_2 \dotplus D_6, \tag{5.3}$$

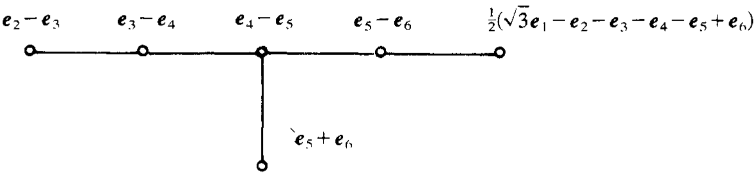
$$D_8 \supset D_3 \dotplus D_5. \tag{5.4}$$

A final word is necessary regarding non-semi-simple subalgebras. The algebras E_6 and E_7 both possess maximal non-semi-simple subalgebras as given in table 12*a* of Dynkin (1957a, p 151). They correspond to the embeddings

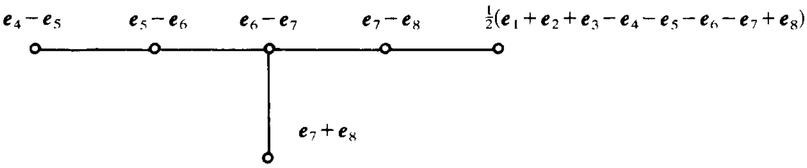
$$E_6 \supset D_1 \dotplus D_5, \tag{5.5}$$

$$E_7 \supset D_1 \dotplus E_6, \tag{5.6}$$

where D_1 is the one-dimensional Lie algebra, neither simple nor semi-simple, of the Lie group SO(2) which is isomorphic to U(1). These embeddings provide labelling schemes. In the case of E_6 the labelling scheme corresponding to (5.5) is



which is not natural by virtue of the presence of the term in $\frac{1}{2}\sqrt{3}e_1$. This situation may be improved by embedding the one-dimensional root space, V, of U(1) in a three-dimensional space W. This yields the simple root system



which is essentially that given by Carter (1972, p 49). Although this is associated in an obvious way with the embedding (5.5), it should be stressed that it is not quite natural, since the natural label for the one and only positive root of D_1 is simply e_1. The use of (5.6) suffers from the same drawback.

A final word of warning is necessary regarding the relationship between the results presented here and those of Wyborne and Bowick (1977). The natural labels based on $G_2 \supset A_2$, $E_6 \supset A_1 \dotplus A_5$, $E_7 \supset A_7$ and $E_8 \supset A_8$ are not quite those of these authors. The difference lies, in the case of the last two groups, in the fact that the natural labelling is based on a lexicographic ordering in W of highest roots and weights. This results in labels, $\boldsymbol{\lambda}$, for representations of E_7 and E_8 which are the contragradient with respect to A_7 and A_8 of the labels, $\boldsymbol{\Lambda}$, used by Wybourne and Bowick, so that for E_7

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_7) = (\Lambda_1, \Lambda_1 - \Lambda_7, \Lambda_1 - \Lambda_6, \ldots, \Lambda_1 - \Lambda_2),$$

whilst for E_8

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_8) = (\Lambda_1, \Lambda_1 - \Lambda_8, \Lambda_1 - \Lambda_7, \ldots, \Lambda_1 - \Lambda_2).$$

In the case of E_6, the distinction between their label $(m_1m_2m_3m_4m_5 : m)$ and the natural label, $\boldsymbol{\lambda}$, is trivial: namely

$$\boldsymbol{\lambda} = (\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5\lambda_6) = (m\ m_1m_2m_3m_4m_5).$$

However, for G_2 the natural label, $\boldsymbol{\lambda}$, is not that introduced first by Racah (1949) and used in the form (u_1, u_2) with $u_1 \geqslant u_2 \geqslant 0$ by Wybourne and Bowick. The relationship between the labels is

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = (u_1 + u_2, u_2)$$

with $\lambda_1 \geqslant 2\lambda_2 \geqslant 0$. The use of the natural label, $\boldsymbol{\lambda}$, circumvents the statement made by Wybourne and Bowick (1977) regarding (5.1) that 'the situation is slightly different for G_2'. The important relation (5.1) does apply to the restriction from G_2 to A_2 if the natural labelling scheme is adopted.

Acknowledgments

One of us (RCK) is pleased to acknowledge the hospitality extended to him in the Mathematics Department of Auckland University where some of this work was carried out and to express his gratitude to the British Council for a Commonwealth Travel Fellowship.

References

Bourbaki N 1968 *Éléments de Mathématique* ch 4–6 (Paris: Hermann)
Cartan E 1894 *Sur la Structure des Groupes de Transformation Finis et Continus* (Paris: Nany)
Carter R W 1972 *Simple Groups of Lie Type* (London: Wiley-Interscience)
Dynkin E B 1957a *Amer. Math. Soc. Transl. Ser. 2* **6** 111–244
—— 1957b *Amer. Math. Soc. Transl. Ser. 2* **6** 245–378
—— 1962 *Amer. Math. Soc. Transl. Ser. 1* **9** 328–469
Gruber B and Samuel M T 1975 in *Group Theory and its Applications* vol III ed E M Loebl (New York: Academic Press) pp 95–141
King R C and Plunkett S P O 1976 *J. Phys. A: Math. Gen.* **9** 863–97
Littlewood D E 1940 *The Theory of Group Characters* (London: Oxford University Press)
Murnaghan F D 1938 *Theory of Group Representations* (Baltimore: Johns Hopkins Press)
—— 1958 *The Orthogonal and Symplectic Groups* A13 (Dublin: Institute for Advanced Studies)
Patera J and Sankoff D 1972 *Branching Rules for Representations of Simple Lie Algebras* CRM-167 (Montréal: Montréal University Press)
Racah G 1949 *Phys. Rev.* **76** 1352–65
—— 1964 in *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gursey (New York: Gordon and Breach) pp 1–36

Speiser D 1964 in *Group Theoretical Concepts and Methods in Elementary Particle Physics* ed F Gursey (New York: Gordon and Breach) pp 237–46
Wan Z-X 1975 *Lie Algebras* (Oxford: Pergamon Press)
Weyl H 1926 *Math. Z.* **24** 377–95
—— 1939 *The Classical Groups* (Princeton: Princeton University Press)
Wybourne B G 1970 *Symmetry Principles and Atomic Spectroscopy* (New York: Wiley-Interscience)
—— 1974 *Classical Groups for Physicists* (New York: Wiley-Interscience)
—— 1979 *Aust. J. Phys.* **32** 417–26
Wybourne B G and Bowick M J 1977 *Aust. J. Phys.* **30** 259–86